Discrete Mathematics, Probability Distributions and Particle Mechanics

A Form 6 Summary, with Applications to Past Examinations

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Acknowledgements

We express our thanks to Dipchand Bahall, and Hassina Ramasra for their guidance while pursuing this subject matter as secondary school students. Lastly, we recognize Deepak Lall, Leon Kalicharan and Raphael Ramlal for support and encouragement in this undertaking.

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Preface

This draft contains notes and solutions to modules one and two. We plan to release the final version, including module three notes and solutions, later this year. Check out the github for more information or this website for downloads.

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Part I

Introduction

Part II

Notes and Definitions

Chapter 1

Module 1: Discrete Mathematics

1.1 Assignment Models

Definition 1.1.1. The **Hungarian Algorithm** is a process that can be used to assign workers of a project to tasks in order minimize a parameter of the project (time, cost, etc).

Definition 1.1.2. The algorithm follows a series of steps as follows:

- **Step 1, Squaring the Matrix**: Ensure that the matrix under question is square (of the form $n \times n$). If the matrix is not square, add the necessary rows or columns such that the new elements are equal to each other and are greater than or equal to the maximum element of the original matrix.

- **Step 2, Reducing Rows**: From each row, find the minimum element and subtract it from all other elements in that row.

- **Step 3, Reducing Columns**: From each column, find the minimum element and subtract it from all other elements in that column.

- **Step 4, Shading the Zeroes**: With the minimum number of horizontal and vertical lines, cover all the zeroes in the matrix. If the total number of lines used is equal to n, there exists a minimum solution among the zeroes in the matrix. If the number of lines is less than n, proceed to Step 5.

- **Step 5, Making New Zeroes**: From all the unshaded elements (those not covered by a line), identify the minimum element, k. Subtract this value from all the unshaded elements and add twice this value ($2k$) to all of the values that are shaded twice. With this new matrix, Return to Step 4. Repeat as necessary.

When a solution exists, go through the zeroes of the matrix by hand to find the optimal solution.

Note 1.1.1. The Hungarian Algorithm can also be used to maximize a project parameter. The steps are exactly the same, however, the signs of all the matrix elements must be flipped.

1.2 Graph Theory and Critical Analysis

Definition 1.2.1. Vertex: The Vertex is the fundamental unit from which graphs are formed. It may also be referred to as a node. P, S and R are examples of vertices in Fig. 1.1

Definition 1.2.2. Edge: In an undirected graph, an unordered pair of nodes that signify a line joining two nodes are said to form an edge. For a directed graph, the edge is an ordered pair of nodes. An edge may also connect a node to itself. The line connecting Q to S in Fig. 1.1 is an example of an edge.

Definition 1.2.3. Path: A path in a graph is a sequence of edges which joins multiple vertices, in which all vertices (and edges) are distinct. The sequence $P \to R \to T$ in Fig. 1.1 is an example of a path.

Definition 1.2.4. Degree: The degree of a vertex in a graph is the number of edges that are incident to the vertex. A loop is counted as two incident edges. For example, the degree of T in Fig. 1.1, $\deg(T)$, is 3.

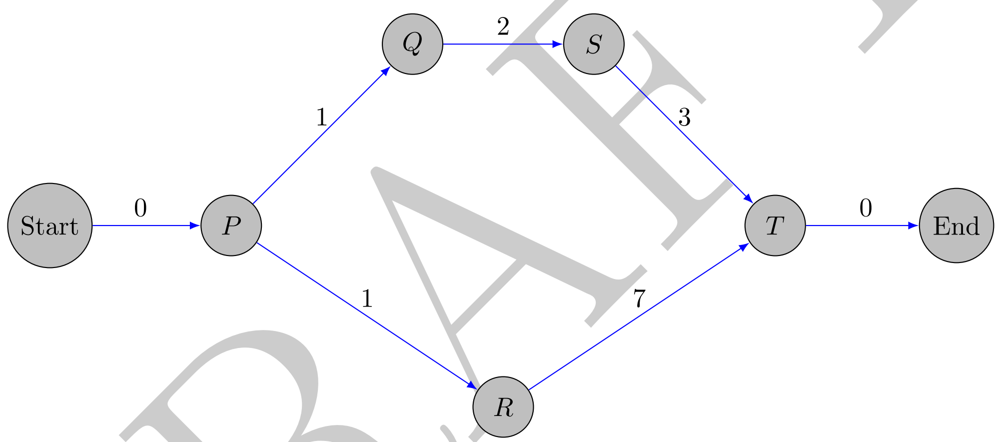

Figure 1.1: Example of an Activity Network.

Definition 1.2.5. Activity Network: An Activity Network is a diagram which shows the order in which activities must be completed throughout a project. If we consider the vertices of Fig. 1.1 to be activities of an arbitrary project, we can see that certain activities must be completed for another to begin. For example, Q can only begin if P is completed and similarly, T can begin if and only if both R and S are complete.

Definition 1.2.6. Float Time: The float time of an activity in an activity network is the time difference between the latest start time of the activity and the earliest start time of the activity.

Definition 1.2.7. Critical Path: A critical path is a path which joins activities in an Activity Network whose Float Time is 0. It is the path which determines the minimum time for some set of operations to be completed. To find the critical path in an activity network, it is useful to construct a table with the activities and their Earliest and Latest Start times. From Table 1.1, we can see that the critical path of Fig. 1.1 is Start $\to P \to R \to T \to$ End.

Definition 1.2.8. Earliest Start Time: To find the earliest start times, you work from the start of the activity network to the end. To find the earliest start time of some activity K_n, you add the weight of the preceding activity to the earliest start time of the preceding activity itself. If K_n has more than one predecessor (for example, T in Fig. 1.1 is preceded by both S and R), you take the largest value obtained. This process if repeated until the end of the Activity Network.

Definition 1.2.9. Latest Start Time: To find the latest start times, you work from the end of the activity network to the start. The latest start time of the final activity in an activity network is equal to the earliest start time of that activity. To find the latest start time of some activity K_j, you subtract the weight of activity K_j from the latest start time of its succeeding activity. If K_j has more than one succeeding activity (for example, P in Fig. 1.1 is succeeded by both Q and R), you take the smallest value obtained. This process is repeated until the start of the Activity Network.

Activity	Earliest Start Time	Latest Start Time	Float Time
P	0	0	0
Q	1	3	2
S	3	5	2
R	1	1	0
T	8	8	0

Table 1.1: Start Times of Fig. 1.1.

1.3 Logic and Boolean Algebra

Definition 1.3.1. Proposition: A proposition is a declarative statement which is either True (denoted by T or 1) or False (denoted by F or 0).

Definition 1.3.2. Propositional Logical Symbols

Symbol	Name	Read as
$\wedge$	Conjunction	And
$\vee$	Disjunction	Or
$\sim$	Negation	Not
$\Rightarrow$	Conditional	If ... then ...
$\Leftrightarrow$	Bi-conditional	If and only if; iff

Definition 1.3.3.

Suppose we have the proposition $\boldsymbol{p \Rightarrow q}$, we make the following three definitions:

Definition 1.3.4. Inverse: $\boldsymbol{\sim p \Rightarrow \sim q}$

Definition 1.3.5. Converse: $\boldsymbol{q \Rightarrow p}$

Definition 1.3.6. Contrapositive: $\boldsymbol{\sim q \Rightarrow \sim p}$

p	$\sim p$
T	F
F	T

Table 1.2: Negation

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 1.3: Conjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 1.4: Disjunction

p	q	$p \implies q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 1.5: Conditional

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 1.6: Bi-conditional

Definition 1.3.7. Tautology: A tautology is a proposition which is true in every possible interpretation.

Definition 1.3.8. Contradiction: A contradiction is a proposition which is false in every possible interpretation.

1.3.1 Laws of Boolean Algebra

Consider the propositional variables $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$

Law 1.3.1. Annulment

$$p \wedge F = F \tag{1.3.1}$$
$$p \vee T = T \tag{1.3.2}$$

Law 1.3.2. Identity

$$p \wedge T = p \tag{1.3.3}$$
$$p \vee F = p \tag{1.3.4}$$

Law 1.3.3. Idempotent

$$p \vee p = p \tag{1.3.5}$$
$$p \wedge p = p \tag{1.3.6}$$

Law 1.3.4. Complement

$$p \vee \sim p = T \tag{1.3.7}$$
$$p \wedge \sim p = F \tag{1.3.8}$$

Law 1.3.5. Double Negation

$$\sim(\sim \boldsymbol{p}) = \boldsymbol{p} \tag{1.3.9}$$

Law 1.3.6. De Morgan's

$$\sim(\boldsymbol{p} \wedge \boldsymbol{q}) = \sim \boldsymbol{p} \vee \sim \boldsymbol{q} \tag{1.3.10}$$
$$\sim(\boldsymbol{p} \vee \boldsymbol{q}) = \sim \boldsymbol{p} \wedge \sim \boldsymbol{q} \tag{1.3.11}$$

Law 1.3.7. Associative

$$(\boldsymbol{p} \vee \boldsymbol{q}) \vee \boldsymbol{r} = \boldsymbol{p} \vee (\boldsymbol{q} \vee \boldsymbol{r}) \tag{1.3.12}$$
$$(\boldsymbol{p} \wedge \boldsymbol{q}) \wedge \boldsymbol{r} = \boldsymbol{p} \wedge (\boldsymbol{q} \wedge \boldsymbol{r}) \tag{1.3.13}$$

Law 1.3.8. Commutative

$$\boldsymbol{p} \wedge \boldsymbol{q} = \boldsymbol{q} \wedge \boldsymbol{p} \tag{1.3.14}$$
$$\boldsymbol{p} \vee \boldsymbol{q} = \boldsymbol{q} \vee \boldsymbol{p} \tag{1.3.15}$$

Law 1.3.9. Distributive

$$\boldsymbol{p} \wedge (\boldsymbol{q} \vee \boldsymbol{r}) = (\boldsymbol{p} \wedge \boldsymbol{q}) \vee (\boldsymbol{p} \wedge \boldsymbol{r}) \tag{1.3.16}$$
$$\boldsymbol{p} \vee (\boldsymbol{q} \wedge \boldsymbol{r}) = (\boldsymbol{p} \vee \boldsymbol{q}) \wedge (\boldsymbol{p} \vee \boldsymbol{r}) \tag{1.3.17}$$

Law 1.3.10. Absorptive

$$\boldsymbol{p} \wedge (\boldsymbol{p} \vee \boldsymbol{q}) = \boldsymbol{p} \tag{1.3.18}$$
$$\boldsymbol{p} \vee (\boldsymbol{p} \wedge \boldsymbol{q}) = \boldsymbol{p} \tag{1.3.19}$$

1.3.2 Logic Circuits

Definition 1.3.9. Logic Circuits can be used to represent Boolean Expressions with switches and wires. Switches are usually defined with logic propositions where a truth value of 1 will indicate a closed switch whereas a truth value of 0 represents an open switch.

Definition 1.3.10. Switching Circuits:

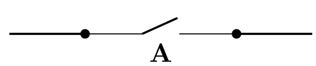

Figure 1.2: Preposition **A**

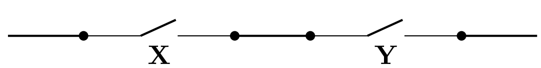

Figure 1.3: Preposition (**X** $\wedge$ **Y**)

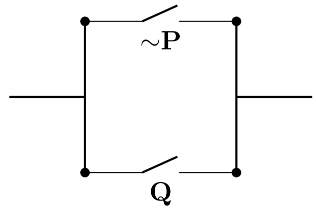

Figure 1.4: Preposition (∼**P** ∨ **Q**)

Chapter 2

Module 2: Probability and Distributions

2.1 Counting Principles

Definition 2.1.1. If event K_i can occur in p ways and event K_j can occur in q ways, the number of ways that K_i **and** K_j can occur is,

$$p \times q. \tag{2.1.1}$$

This can be extended to any number of events. This is known as the **Multiplication Rule**.

Definition 2.1.2. If event K_i can occur in p ways and event K_j can occur in q ways, the number of ways that K_i **or** K_j can occur is,

$$p + q. \tag{2.1.2}$$

This can be extended to any number of events. This is known as the **Addition Rule**.

Definition 2.1.3. A **combination** is a group of elements chosen from a set of objects. The order of the elements does not matter.

Definition 2.1.4. The number of combinations of r elements from a set of n **distinct** elements is defined as,

$${}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}\,. \tag{2.1.3}$$

Definition 2.1.5. A **permutation** is a group of elements chosen from a set of objects where the order of the elements **does** matter.

Definition 2.1.6. The number of permutations of r elements from a set of n **distinct** elements is defined as,

$${}^{n}P_{r} = \frac{n!}{(n-r)!}\,. \tag{2.1.4}$$

Definition 2.1.7. The number of permutations of n elements, not all distinct, is defined as,

$$\frac{n!}{k_1!\,k_2!\ldots k_r!}\,, \tag{2.1.5}$$

where $\sum_{i=1}^{r} k_i = n$, and k_i is the number of non-distinct elements of type i.

Definition 2.1.8. The number of permutations of size r out of n elements, with repetition, is defined as,

$$n^r \tag{2.1.6}$$

2.2 Probability Theory

Axiom 2.2.1. Consider an experiment whose sample space is S. The **probability of the event** E, denoted as $P(E)$, is a number that satisfies the following three axioms

(1) $0 \leq P(E) \leq 1$,

(2) $P(S) = 1$,

(3) For any sequence of mutually exclusive events $E_1, E_2, ..., E_n$.
$P\left(\cup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$,
Eg. $P(E_1 \cup E_2) = P(E_1) + P(E_2)$.

Definition 2.2.1. Furthermore, we define,

$$P(\text{event E occurs}) = \frac{\text{no. of times A can occur}}{\text{total number of outcomes}}. \tag{2.2.1}$$

Definition 2.2.2. The probability of the **complement** of event A, denoted as $P(A^c)$ or $P(\bar{A})$ or $P(A')$, is defined as

$$P(\bar{A}) = 1 - P(A). \tag{2.2.2}$$

Proposition 2.2.1. Some useful propositions can be easily derived using the above and drawing your own diagrams

$$P(A \cup B) = P(A) + P(B) - P(A \cap B), \tag{2.2.3}$$
$$P(A^c \cap B) = P(B) - P(A \cap B), \tag{2.2.4}$$
$$P(A^c \cap B^c) = P(A \cup B)^c, \tag{2.2.5}$$
$$P(A^c \cup B^c) = P(A \cap B)^c. \tag{2.2.6}$$

Definition 2.2.3. Two events A and B are said to be **mutually exclusive** if and only if

$$P(A \cap B) = 0. \tag{2.2.7}$$

Applying Eq [2.2.3], this implies

$$P(A \cup B) = P(A) + P(B). \tag{2.2.8}$$

Definition 2.2.4. Two events A and B are said to be **independent** if and only if

$$P(A \cap B) = P(A) \times P(B). \tag{2.2.9}$$

Definition 2.2.5. For events A and B, the **conditional probability** of A given B has occurred, denoted by $P(A|B)$, is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}. \tag{2.2.10}$$

2.3 Discrete Random Variables

Definition 2.3.1. A **discrete random variable**, X, can take on at most a countable number of possible values. We define its *probability mass* function, $P(x)$ by

$$P(x) = P(X = x). \tag{2.3.1}$$

Proposition 2.3.1. If X is a discrete random variable with probability mass function $P(x)$, then we know from Axiom [**(1)**] that

$$0 \le P(X = x) \le 1 \, \forall x. \tag{2.3.2}$$

and from Axiom [**(2)**] and Axiom [**(3)**] that

$$\sum_{\forall i} P(X = x_i) = 1. \tag{2.3.3}$$

Definition 2.3.2. If X is a discrete random variable, the **expectation value**, **mean** or **first moment of** X, $E[X]$, is defined by

$$E[X] = \sum_{\forall i} x_i P(X = x_i). \tag{2.3.4}$$

Definition 2.3.3. If X is a discrete random variable, the **second moment of** X is defined by

$$E[X^2] = \sum_{\forall i} x_i^2 P(X = x_i). \tag{2.3.5}$$

Definition 2.3.4. If X is a random variable with mean μ, then the **variance**, $Var[X]$, is defined by

$$\mathrm{Var}[X] = E[(X - \mu)^2], \tag{2.3.6}$$
$$= E[X^2] - (E[X])^2 = E[X^2] - \mu^2. \tag{2.3.7}$$

2.4 Discrete Uniform Distribution

Definition 2.4.1. A discrete random variable X is said to follow a **discrete uniform distribution** if X can take a finite number of values which are observed with equal probabilities. Therefore, the probability mass function, P, of X which can take n possible values, is given by

$$P(X = x_i) = \frac{1}{n}. \tag{2.4.1}$$

Note 2.4.1. In the discrete case, the notation $X \sim \mathrm{Unif}(a, b)$ or $X \sim \mathrm{U}(a, b)$ implies that X can take **integer** values x such that $a \le x \le b$. Here, the total number of values X can take is given by

$$n = b - a + 1, \tag{2.4.2}$$

Therefore the p.m.f of X can be given by

$$P(X = x) = \frac{1}{b - a + 1} \qquad a \le x \le b. \tag{2.4.3}$$

The *expected value (mean)* and *variance* of $X \sim \mathrm{U}(a,b)$ are given by:

$$E[X] = \frac{a+b}{2}, \tag{2.4.4}$$

$$\mathrm{Var}[X] = \frac{(b-a+1)^2-1}{12}. \tag{2.4.5}$$

2.5 Binomial Distribution

Definition 2.5.1. A discrete random variable X is said to follow a **binomial distribution** with parameters n and p, $X \sim \mathrm{Bin}(n,p)$, if its probability mass function, P, is given by

$$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x \in (0,1,2,...,n). \tag{2.5.1}$$

Note 2.5.1. The *expected value (mean)* and *variance* of X are given by:

$$E[X] = np, \tag{2.5.2}$$

$$\mathrm{Var}[X] = np(1-p), \tag{2.5.3}$$

$$= npq \;\text{ where } q = (1-p). \tag{2.5.4}$$

Note 2.5.2. There are four conditions that describe a binomial distribution

(i) The experiment consist of a fixed number of trials, n.

(ii) The trials are independent.

(iii) Each trial can be classified as a success or failure.

(iv) The probability of success, p, is constant

2.6 Geometric Distribution

Definition 2.6.1. A discrete random variable X is said to follow a **geometric distribution** with parameter p, $X \sim \mathrm{Geo}(p)$, if its probability mass function, P, is given by

$$P(X=x) = p(1-p)^{x-1} \quad x \in (1,2,...). \tag{2.6.1}$$

Note 2.6.1. The *expected value (mean)* and *variance* of X are given by:

$$E[X] = \frac{1}{p}, \tag{2.6.2}$$

$$\mathrm{Var}[X] = \frac{1-p}{p^2}, \tag{2.6.3}$$

$$= \frac{q}{p^2} \;\text{ where } q = (1-p). \tag{2.6.4}$$

Note 2.6.2. The probability of $X > k$ is given as

$$P(X>k) = q^k \tag{2.6.5}$$

Note 2.6.3. There are three conditions that describe a geometric distribution

(i) The trials are independent.

(ii) Each trial can be classified as a success or failure.

(iii) The probability of success, p, is constant

2.7 Poisson Distribution

Definition 2.7.1. A discrete random variable X is said to follow a **poisson distribution** with parameter λ, $X \sim \text{Pois}(\lambda)$, if its probability mass function, P, is given by

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad x \in (0, 1, 2, ...). \tag{2.7.1}$$

Note 2.7.1. The *expected value (mean)* and *variance* of X are given by:

$$E[X] = \lambda, \tag{2.7.2}$$
$$\text{Var}[X] = \lambda. \tag{2.7.3}$$

Note 2.7.2. The Poisson distribution is popular for modeling the number of times an event occurs in an interval of time. There are two conditions that describe a poisson distribution

(i) Events are independent of each other.

(ii) The average rate at which events occur, λ, is independent of any occurrences.

2.8 Poisson Approximation to the Binomial Distribution

Definition 2.8.1. The **Poisson limit theorem** states that a binomially distributed random variable $X \sim \text{Bin}\{n, p\}$ can be approximated by a poisson distribution if the following hold;

(i) $n > 50$,

(ii) $np < 5$.

Under the above conditions, $X \sim \text{Bin}(n, p)$ can be approximated by $X \sim \text{Pois}(np)$, $\lambda = E[X]$.

2.9 Continuous Random Variables

Definition 2.9.1. We say that X is a **continuous random variable**, if there exists a non-negative function f, defined for all real $x \in (-\infty, \infty)$, having the property that, for any set of real numbers B,

$$P(X \in B) = \int_B f(x).dx. \tag{2.9.1}$$

The function f is called the *probability density function* of the random variable X.

Proposition 2.9.1. Since X must assume some value, f must satisfy,

$$1 = P(X \in (-\infty, \infty)) = \int_{-\infty}^{\infty} f(x).dx. \tag{2.9.2}$$

Note 2.9.1. If $B = [a, b]$, then

$$P(X \in B) = P(a \leq X \leq b), \tag{2.9.3}$$
$$= \int_{a}^{b} f(x).dx. \tag{2.9.4}$$

Proposition 2.9.2. For any continuous random variable X and real number a,

$$P(X = a) = 0. \tag{2.9.5}$$

Note 2.9.2. This implies

$$P(X \leq a) = P(X < a). \tag{2.9.6}$$

Definition 2.9.2. If X is a continuous random variable, the **expectation (mean)** of X is given by,

$$E[X] = \int_{-\infty}^{\infty} x f(x).dx. \tag{2.9.7}$$

Definition 2.9.3. If X is a continuous random variable, the **variance** of X is given by,

$$\mathrm{Var}[X] = E[X^2] - E[X]^2, \tag{2.9.8}$$
$$= \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x - \left(\int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x\right)^2. \tag{2.9.9}$$

Law 2.9.1. In the above formula for variance of a continuous random variable, the **Law of the Unconscious Statistician (LOTUS)** is used. This law states that if X is a continuous random variable with probability density function f, then for any function g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x).dx. \tag{2.9.10}$$

The law also extends to the discrete case. If X is a discrete random variable, then by **LOTUS**,

$$E[g(X)] = \sum_{\forall i} g(x_i) P(X = x_i). \tag{2.9.11}$$

Definition 2.9.4. For a continuous random variable X with p.d.f f, the **cumulative distribution function**, F, is defined as,

$$F(x) = P(X \leq x), \tag{2.9.12}$$

Using Eq [2.9.1], where $B = (-\infty, x]$,

$$F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t, \tag{2.9.13}$$

Using the **Fundamental Law of Calculus**, then

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x} F(x). \tag{2.9.14}$$

Note 2.9.3. This implies

$$P(a < X \leq b) = F(b) - F(a). \tag{2.9.15}$$

Note 2.9.4. A useful result for $P(X > k)$ is

$$P(X > k) = 1 - F(k). \tag{2.9.16}$$

Proposition 2.9.3. The c.d.f can be used to calculate the median and quartiles of a distribution. If a continuous random variable X has c.d.f F, then

$$F(M) = 0.5, \tag{2.9.17}$$
$$F(LQ) = 0.25, \tag{2.9.18}$$
$$F(UQ) = 0.75. \tag{2.9.19}$$

where M denotes the median, LQ denotes the lower quartile and UQ denotes the upper quartile.

2.10 Normal Distribution

Definition 2.10.1. A continuous random variable X is said to follow a **normal distribution** with parameters μ and σ^2, $X \sim \mathrm{N}(\mu, \sigma^2)$, if its probability density function, f, is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \qquad -\infty < x < \infty \tag{2.10.1}$$

The parameters μ and σ^2 represent the expected value (mean) and variance respectively.

Note 2.10.1. Normal random variables have important properties

(i) If X is normally distributed with parameters μ and σ^2, then $Y = aX+b$ is normally distributed with parameters $a\mu + b$ and $a^2\sigma^2$ respectively.

(ii) The previous property implies that if X is normally distributed with parameters μ and σ^2, then $Z = \frac{(X-\mu)}{\sigma}$ is normally distributed with parameters 0 and 1. We call Z a *standard* or *unit* normal random variable.

2.11 Normal Approximation to the Poisson Distribution

Definition 2.11.1. A poisson distributions $X \sim \mathrm{Pois}(\lambda)$ where $\lambda > 15$, can be approximated by a normal distribution, $X \sim \mathrm{N}(\lambda, \sqrt{\lambda})$.

Note 2.11.1. To use this approximation, we must note that because the poisson is a discrete integer-valued random variable and the normal is a continuous random variable, we must write

$$P(X = a) = P\left(a - \frac{1}{2} < X < a + \frac{1}{2}\right), \tag{2.11.1}$$

$$P(X \leq a) = P\left(X < a + \frac{1}{2}\right), \tag{2.11.2}$$

$$P(X \geq a) = P\left(X > a - \frac{1}{2}\right), \tag{2.11.3}$$

$$P(X < a) = P\left(X < a - \frac{1}{2}\right), \tag{2.11.4}$$

$$P(X > a) = P\left(X > a + \frac{1}{2}\right), \tag{2.11.5}$$

$$P(a \leq X < b) = P\left(a - \frac{1}{2} < X < b - \frac{1}{2}\right), \tag{2.11.6}$$

$$P(a < X \leq b) = P\left(a + \frac{1}{2} < X < b + \frac{1}{2}\right), \tag{2.11.7}$$

$$P(a \leq X \leq b) = P\left(a - \frac{1}{2} < X < b + \frac{1}{2}\right). \tag{2.11.8}$$

This is called the **continuity correction**.

2.12 Expectation and Variance

Proposition 2.12.1. Define functions g and f with $a, b, c \in \mathbb{R}$. For **ANY** random variables (discrete or continuous) X and Y, the following properties hold

$$E[aX + b] = aE[X] + b, \tag{2.12.1}$$
$$E[X + Y] = E[X] + E[Y], \tag{2.12.2}$$
$$E[ag(X) + bf(Y) + c] = aE[g(X)] + bE[f(Y)] + c, \tag{2.12.3}$$
$$\text{Var}[ag(X) + b] = a^2\text{Var}[g(X)]. \tag{2.12.4}$$

If X and Y are **independent** random variables, then the following properties also hold

$$E[XY] = E[X]\ E[Y], \tag{2.12.5}$$
$$E[g(X)f(Y)] = E[g(X)]\ E[f(Y)], \tag{2.12.6}$$
$$\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y]. \tag{2.12.7}$$

Note 2.12.1. If $E[XY] = E[X]\ E[Y]$ or $\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y]$, this does not imply X and Y are independent.

2.13 χ^2 goodness-of-fit test

This test is used to determine whether a random variable can be modeled with a given distribution. One can conduct a χ^2 test following the simple procedure:

1) Define the null hypothesis, H_0, and alternative hypothesis, H_1:

$$H_0 \quad : \quad \text{The data follows the given distribution.} \tag{2.13.1}$$
$$H_1 \quad : \quad \text{The data does not follow the given distribution.} \tag{2.13.2}$$

2) Pick a level of significance α
3) Identify critical region: Reject H_0 if $\chi^2_{\text{calc}} > \chi^2_{\alpha}(\nu)$
4) $\chi^2_{\text{calc}} = \sum \frac{(O-E)^2}{E}$ where O is the observed data, and E is the expected frequency. We can compute the expected frequency using $E_{ij} = \frac{n_i \times n_j}{N}$, where E_{ij} represents the expected frequency of the value in the i^{th} row and j^{th} column, n_i represents the sum of values in the i^{th} row, n_j represents the sum of values in the j^{th} column, and N represents the total sample size.
5) Decision

Note 2.13.1. χ^2_{calc} is not valid if $E < 5$. When this occurs we combine classes to make all values of $E > 5$.

Note 2.13.2. ν is the number of degrees of freedom in the test. It is calculated as follows,

$$\nu = \text{Number of classes - Number of restrictions}\,. \tag{2.13.3}$$

The number of restrictions in a test depends on the distribution that the test is considering. The number of restrictions is as follows:

- Uniform Distribution:
  - 1 restriction (Sum of Observed Frequencies must equal Sum of Expected Frequencies)
- Distribution in a Given Ratio:
  - 1 restriction (Sum of Observed Frequencies must equal Sum of Expected Frequencies)
- Binomial Distribution:
  - 1 restriction if chance of success, p, is known (Sum of Observed Frequencies must equal Sum of Expected Frequencies)
  - 2 restrictions if chance of success, p, is unknown (Sum of Observed Frequencies must equal Sum of Expected Frequencies AND $p = \frac{\bar{x}}{n}$)
- Poisson Distribution:
  - 1 restriction if average rate, λ, is known (Sum of Observed Frequencies must equal Sum of Expected Frequencies)
  - 2 restrictions if average rate, λ, is unknown (Sum of Observed Frequencies must equal Sum of Expected Frequencies AND $\lambda = \bar{x}$)
- Geometric Distribution:

  - 1 restriction if chance of success, p, is known (Sum of Observed Frequencies must equal Sum of Expected Frequencies)
  - 2 restrictions if chance of success, p, is unknown (Sum of Observed Frequencies must equal Sum of Expected Frequencies AND $p = \frac{1}{\bar{x}}$)

- Normal Distribution:
  - 1 restriction if both mean, μ, and variance, σ^2 are known (Sum of Observed Frequencies must equal Sum of Expected Frequencies)
  - 2 restrictions if either mean, μ, or variance, σ^2 is unknown (Sum of Observed Frequencies must equal Sum of Expected Frequencies AND $\mu = \bar{x}$ OR $\sigma^2 = \hat{\sigma}^2$)
  - 3 restrictions if both mean, μ, and variance, σ^2 are unknown (Sum of Observed Frequencies must equal Sum of Expected Frequencies AND $\mu = \bar{x}$ AND $\sigma^2 = \hat{\sigma}^2$)

The restrictions can be summarized as follows:

Distribution	**Parameters**	**Number of Restrictions**
Uniform Distribution	-	1
Distribution in a Given Ratio	-	1
Binomial Distribution	p known	1
	p unknown	2
Poisson Distribution	λ known	1
	λ unknown	2
Geometric Distribution	p known	1
	p unknown	2
Normal Distribution	μ and σ^2 known	1
	either μ or σ^2 unknown	2
	both μ and σ^2 unknown	3

Table 2.1: : Restrictions in a χ^2 goodness-of-fit test.

Chapter 3

Module 3: Particle Mechanics

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Chapter 4

Problem Solving Strategies

Part III

Solutions

Chapter 5

2005

5.1 Module 1: Managing Uncertainty

1. (a) (i) We can construct the following truth table for the statement $p \vee q$.

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

Table 5.1: Truth table of $p \vee q$.

(ii) We can complete the truth table in the following way.

p	q	$\sim p$	$p \implies q$	$\sim p \wedge q$	$(p \implies q) \iff (\sim p \wedge q)$
0	0	1	1	0	0
0	1	1	1	1	1
1	0	0	0	0	1
1	1	0	1	0	0

Table 5.2: Truth table of $(p \implies q) \iff (\sim p \wedge q)$.

(iii) a) Since the truth values of Table 5.1 and Table 5.2 are not the same, the expressions $p \vee q$ and $(p \implies q) \iff (\sim p \wedge q)$ are not logically equivalent.

b) From Def. 1.3.7, we can see that $((p \implies q) \iff (\sim p \wedge q))$ is **not a tautology** because the truth values are not all 1.

(b) (i) If x is the proposition "I do Statistical Analysis" and y is the proposition "I do Applied Mathematics", then $x \wedge \sim y$ can be expressed as "I do Statistical Analysis and I do not do Applied Mathematics."

(ii) The proposition "I do Statistical Analysis so I will do Applied Mathematics" can be interpreted in logical language as "If I do Statistical Analysis, then I do Applied

Mathematics also". This can be expressed as:

$$\boldsymbol{x} \implies \boldsymbol{y}\,. \tag{5.1.1}$$

(c) Let $\boldsymbol{A} = (\boldsymbol{l} \wedge \boldsymbol{m} \wedge \boldsymbol{n}) \vee (\boldsymbol{l} \wedge \boldsymbol{m} \wedge \sim \boldsymbol{n})$.
Let $\boldsymbol{B} = \boldsymbol{l} \wedge \boldsymbol{m}$.
A then becomes,

$$\boldsymbol{A} \equiv (\boldsymbol{B} \wedge \boldsymbol{n}) \vee (\boldsymbol{B} \wedge \sim \boldsymbol{n})\,. \tag{5.1.2}$$

Using the Distributive Law (Def. 1.3.9) in reverse, A becomes,

$$\boldsymbol{A} \equiv \boldsymbol{B} \wedge (\boldsymbol{n} \vee \sim \boldsymbol{n})\ \,. \tag{5.1.3}$$

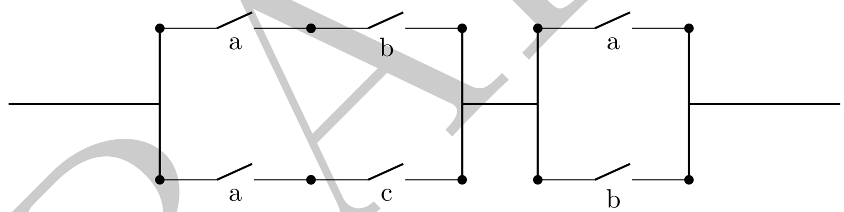
Using the Complement Law (Def. 1.3.4) and Identity Law (Def. 1.3.2), A is now,

$$\begin{aligned} \boldsymbol{A} &\equiv \boldsymbol{B} \wedge \boldsymbol{T} \\ &\equiv \boldsymbol{B}\,. \end{aligned} \tag{5.1.4}$$

Therefore,

$$(\boldsymbol{l} \wedge \boldsymbol{m} \wedge \boldsymbol{n}) \vee (\boldsymbol{l} \wedge \boldsymbol{m} \wedge \sim \boldsymbol{n}) \equiv \boldsymbol{l} \wedge \boldsymbol{m}\,. \tag{5.1.5}$$

(d) See the switching circuit of $((\boldsymbol{a} \wedge \boldsymbol{b}) \vee (\boldsymbol{a} \wedge \boldsymbol{c})) \wedge (\boldsymbol{a} \vee \boldsymbol{b})$.



2. **(a)** The objective function can be given as,

$$P = 300x + 550y\,. \tag{5.2.1}$$

(b) The 4 constraints to this problem are,

$$\begin{aligned} \text{Constraint of number of pine cabinets: } & x \geq 0\,, \\ \text{Constraint of number of mahogany cabinets: } & y \geq 0\,, \\ \text{Constraint of time: } & 3x + 5y \leq 30\,, \\ \text{Constraint of cost: } & 100x + 250y \leq 1250\,. \end{aligned} \tag{5.2.2}$$

As it is required to give each constraint in simplest form, the Cost Constraint can be reduced to:

$$100x + 250y \leq 1250 \implies 2x + 5y \leq 25\,. \tag{5.2.3}$$

(c) The feasible region is shaded in Fig. 5.1.

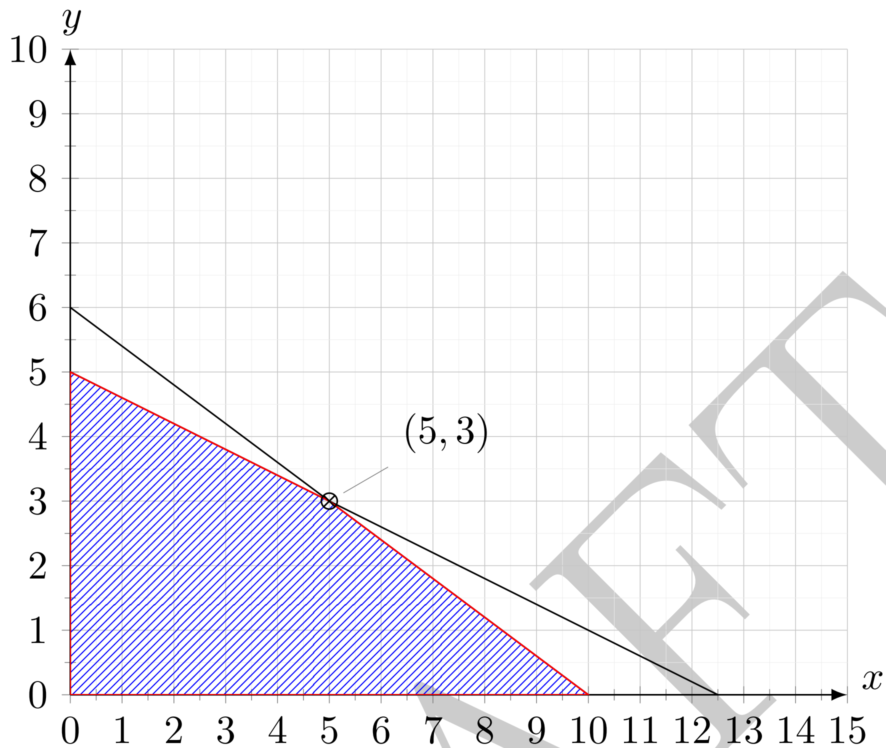


Figure 5.1: Linear Programming Graph.

(d) By performing a Tour of the Vertices (Def. **??**), we can evaluate the objective Profit function.

Vertice	$P = 300x + 550y$
(0, 0)	0
(0, 5)	2750
(5, 3)	2550
(10, 0)	3000

Table 5.3: Tour of Vertices

From Table 5.3, we see that maximum profit P is \$3000 when $x = 10$ and $y = 0$.

(e) The Profit Line method can also be used to determine the maximum value of P. It relies on drawing parallel lines, starting from the origin, which have the gradient of the objective function. This line is then drawn at different points on the x or y axes. The furthest point on the feasible region which the Profit Line touches (just before leaving the feasible region) is the point which will yield the maximum value of the objective function.

5.2 Module 2: Probability and Distributions

3. **(a)** We are given the probability density function, $f(x)$, of a continuous random variable X.

We can use the property that probabilities must sum to one, Prop. 2.9.1 for continuous random variables,

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1\,. \tag{5.3.1}$$

Performing the integral requires us to split the integral into three regions,

$$\int_{-\infty}^{1} f(x)\mathrm{d}x + \int_{1}^{4} f(x)\mathrm{d}x + \int_{4}^{\infty} f(x)\mathrm{d}x = 1\,. \tag{5.3.2}$$

Since we know how the functions are defined in each region, we can now solve for k,

$$\begin{aligned} 0 + \int_{1}^{4} kx^3\mathrm{d}x + 0 &= 1 \\ k \times \left[\frac{x^4}{4}\right]_1^4 &= 1 \\ k \times \left[\frac{4^4}{4} - \frac{1^4}{4}\right] &= 1 \\ k \times \left[\frac{256}{4} - \frac{1}{4}\right] &= 1 \\ \frac{255k}{4} &= 1 \\ \implies k &= \frac{4}{255}\,. \end{aligned} \tag{5.3.3}$$

(b) **(i)** From Note 2.9.1, we know that we must integrate the probabilities in the given region,

$$\begin{aligned} P(2 \leq X) &= P(2 \leq X \leq \infty) \\ &= \int_{2}^{\infty} f(x)\ \mathrm{d}x\,. \end{aligned} \tag{5.3.4}$$

Again, we must now split up the integral over the regions which f is defined,

$$P(2 \leq X) = \int_{2}^{4} f(x)\ \mathrm{d}x + \int_{4}^{\infty} f(x)\ \mathrm{d}x\,. \tag{5.3.5}$$

Finally, we can evaluate,

$$\begin{aligned}
&= \int_2^4 \frac{4x^3}{255}\mathrm{d}x + 0 \\
&= \frac{4}{255} \times \int_2^4 x^3 \mathrm{d}x \\
&= \frac{4}{255} \times \left[\frac{x^4}{4}\right]_2^4 \\
&= \frac{4}{255} \times \left[\frac{4^4}{4} - \frac{2^4}{4}\right] \\
&= \frac{4}{255} \times \left[\frac{256}{4} - \frac{16}{4}\right] \\
&= \frac{4}{255} \times \frac{240}{4} \\
&= \frac{240}{255} = \frac{16}{17}\,. \qquad (5.3.6)
\end{aligned}$$

(ii) From Def. 2.9.2, we know that the expectation of a continuous random variable X is given by,

$$E(X) = \int_{-\infty}^{\infty} x f(x)\mathrm{d}x\,. \qquad (5.3.7)$$

Splitting up the integral as before,

$$E(X) = \int_{-\infty}^{1} x f(x)\mathrm{d}x + \int_1^4 x f(x)\mathrm{d}x + \int_4^{\infty} x f(x)\mathrm{d}x\,, \qquad (5.3.8)$$

allows us to evaluate,

$$\begin{aligned}
E(X) &= 0 + \int_1^4 \frac{4x^4}{255}\mathrm{d}x + 0 \\
&= \frac{4}{255} \times \int_1^4 x^4 \mathrm{d}x \\
&= \frac{4}{255} \times \left[\frac{x^5}{5}\right]_1^4 \\
&= \frac{4}{255} \times \left[\frac{4^5}{5} - \frac{1^5}{5}\right] \\
&= \frac{4}{255} \times \left[\frac{1024}{5} - \frac{1}{5}\right] \\
&= \frac{4}{255} \times \frac{1023}{5} \\
&= \frac{4092}{1275} \approx 3.21\,. \qquad (5.3.9)
\end{aligned}$$

From Def. 2.9.3, we know that the variance is defined via,

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 . \tag{5.3.10}$$

Hence, all that is left to compute is $E(X^2)$. We can do this by using Eq. (2.9.9),

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) \mathrm{d}x . \tag{5.3.11}$$

Again, the integral must be split up appropriately,

$$E(X^2) = \int_{-\infty}^{1} x^2 f(x) \mathrm{d}x + \int_{1}^{4} x^2 f(x) \mathrm{d}x + \int_{4}^{\infty} x^2 f(x) \mathrm{d}x , \tag{5.3.12}$$

so that we can evaluate as follows,

$$\begin{aligned}
E(X^2) &= 0 + \int_1^4 \frac{4x^5}{255} \mathrm{d}x + 0 \\
&= \frac{4}{255} \times \int_1^4 x^5 \mathrm{d}x \\
&= \frac{4}{255} \times \left[\frac{x^6}{6}\right]_1^4 \\
&= \frac{4}{255} \times \left[\frac{4^6}{6} - \frac{1^6}{6}\right] \\
&= \frac{4}{255} \times \left[\frac{4096}{6} - \frac{1}{6}\right] \\
&= \frac{4}{255} \times \frac{4095}{6} \\
&= \frac{16380}{1530} \approx 10.71 .
\end{aligned} \tag{5.3.13}$$

Substituting these values into Eq. (5.3.10), we have,

$$\begin{aligned}
\mathrm{Var}(X) &= E(X^2) - E(X)^2 \\
&= 10.71 - 3.21^2 \\
&= 0.406 .
\end{aligned} \tag{5.3.14}$$

(c) (i) Using Def. 2.9.4, we know that the cumulative distribution function $F(x)$ is defined by,

$$F(x) = \int_{-\infty}^{x} f(t)\ \mathrm{d}t . \tag{5.3.15}$$

Now, we must evaluate this integral over different values of x, corresponding to the intervals which f is defined.

For $x < 1$,

$$
\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(t)\ \mathrm{d}t \\
&= 0\,. \qquad (5.3.16)
\end{aligned}
$$

For $1 \le x < 4$,

$$
\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(t)\ \mathrm{d}t \\
&= \int_{-\infty}^{1} f(t)\ \mathrm{d}t + \int_{1}^{x} f(t)\ \mathrm{d}t \\
&= 0 + \int_{1}^{x} t^3 \mathrm{d}t \\
&= \frac{4}{255} \times \left[\frac{t^4}{4}\right]_1^x \\
&= \frac{4}{255} \times \left[\frac{x^4}{4} - \frac{1^4}{4}\right] \\
&= \frac{1}{255}\left(x^4 - 1\right)\,. \qquad (5.3.17)
\end{aligned}
$$

For $x \ge 4$,

$$
\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(t)\ \mathrm{d}t \\
&= \int_{-\infty}^{1} f(t)\ \mathrm{d}t + \int_{1}^{4} f(t)\ \mathrm{d}t + \int_{4}^{x} f(t)\ \mathrm{d}t \\
&= 0 + \frac{4^4 - 1}{255} + 0 \\
&= 1\,. \qquad (5.3.18)
\end{aligned}
$$

Putting this information together, we have,

$$
F(x) = \begin{cases} 0, & x \le 1 \\ \frac{1}{255}\left(x^4 - 1\right), & 1 \le x < 4 \\ 1, & x \ge 4\,. \end{cases} \qquad (5.3.19)
$$

(ii) From Prop. 2.9.3, know that the lower quartile, LQ, is the value of x such that

$$
F(x = LQ) = 0.25\,. \qquad (5.3.20)
$$

Since the value of $F(LQ)$ is between 0 and 1, we know from our definition of the c.d.f above, Eq. (5.3.19), that $1 \le LQ < 4$. Hence, we can find LQ by equating the following,

$$\begin{aligned}
\frac{1}{255}\left((LQ)^4 - 1\right) &= 0.25 \\
(LQ)^4 - 1 &= \frac{255}{4} \\
(LQ)^4 &= \frac{259}{4} \\
\implies LQ &= \sqrt[4]{\frac{259}{4}} \\
&\approx 2.84 \,. \qquad (5.3.21)
\end{aligned}$$

4. (a) (i) Let X be the discrete random variable representing "the number of adults dissatisfied with their health coverage from a group of 15." We determine that

$$X \sim \mathrm{Bin}(15, 0.05)\,. \qquad (5.4.1)$$

a) From Def. 2.5.1, we know that,

$$P(X = x) = \binom{15}{x}(0.05)^x(1 - 0.05)^{15-x}\,. \qquad (5.4.2)$$

Hence,

$$\begin{aligned}
P(X = 2) &= \binom{15}{2} \times (0.05)^2 \times (1 - 0.05)^{15-2} \\
&= 105 \times (0.05)^2 \times (0.95)^{13} \\
&= 0.135\,. \qquad (5.4.3)
\end{aligned}$$

b) Since X is discrete, we know that,

$$\begin{aligned}
P(X > 2) &= 1 - P(X \leq 2) \\
&= 1 - \left(P(X = 0) + P(X = 1) + P(X = 2)\right)\,. \qquad (5.4.4)
\end{aligned}$$

Using Eq. (5.4.2), we can calculate the individual probabilities,

$$\begin{aligned}
P(X = 0) &= \binom{15}{0} \times (0.05)^0 \times (0.95)^{15-0} \\
&= 1 \times 1 \times (0.95)^{15} \\
&= 0.463\,. \qquad (5.4.5)
\end{aligned}$$

$$\begin{aligned}
P(X = 1) &= \binom{15}{1} \times (0.05)^1 \times (0.95)^{15-1} \\
&= 15 \times 0.05 \times (0.95)^{14} \\
&= 0.366\,. \qquad (5.4.6)
\end{aligned}$$

$$\begin{aligned}
P(X = 2) &= \binom{15}{2} \times (0.05)^2 \times (0.95)^{15-2} \\
&= 15 \times 0.05 \times (0.95)^{13} \\
&= 0.135\,. \qquad (5.4.7)
\end{aligned}$$

Substituting these values into Eq. (5.4.4), we can find that,

$$\begin{aligned} P(X > 2) &= 1 - (P(X = 0) + P(X = 1) + P(X = 2)) \\ &= 1 - (0.463 + 0.366 + 0.135) \\ &= 1 - 0.964 \\ &= 0.036\,. \end{aligned} \tag{5.4.8}$$

(ii) Let X be the discrete random variable representing " the number of adults dissatisfied with their health coverage from a group of 60". Given that $n = 60$ and $p = 0.05$, at first we can say

$$X \sim \text{Bin}(60, 0.05)\,. \tag{5.4.9}$$

However, since

$$\begin{aligned} n &= 60 > 30, \text{and} \\ np &= 60 \times 0.05 = 3 < 5\,, \end{aligned} \tag{5.4.10}$$

we can apply the Poisson approximation to the Binomial distribution, Def. 2.8.1, as follows,

$$X \sim \text{Pois}(np) \equiv X \sim \text{Pois}(3)\,. \tag{5.4.11}$$

Now we can use this to compute $P(X \leq 3)$. Since X is discrete,

$$P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)\,. \tag{5.4.12}$$

For a Poisson random variable, we know that according to Def. 2.7.1,

$$P(X = x) = \frac{3^x \times e^{-3}}{x!}\,. \tag{5.4.13}$$

From Eq. (5.4.13), we need to calculate the following,

$$\begin{aligned} P(X = 0) &= \frac{3^0 \times e^{-3}}{0!} \\ &= \frac{1 \times e^{-3}}{1} \\ &= e^{-3}\,. \end{aligned} \tag{5.4.14}$$

$$\begin{aligned} P(X = 1) &= \frac{3^1 \times e^{-3}}{1!} \\ &= \frac{3 \times e^{-3}}{1} \\ &= 3e^{-3}\,. \end{aligned} \tag{5.4.15}$$

$$\begin{aligned} P(X = 2) &= \frac{3^2 \times e^{-3}}{2!} \\ &= \frac{9 \times e^{-3}}{2} \\ &= \frac{9e^{-3}}{2}\,. \end{aligned} \tag{5.4.16}$$

$$
\begin{aligned}
P(X=3) &= \frac{3^3 \times e^{-3}}{3!} \\
&= \frac{27 \times e^{-3}}{6} \\
&= \frac{9e^{-3}}{2}\,.
\end{aligned}
\tag{5.4.17}
$$

Substituting these values into Eq. (5.4.12), we find,

$$
\begin{aligned}
P(X \leq 3) &= P(X=0) + P(X=1) + P(X=2) + P(X=3) \\
&= e^{-3} \times \left(1 + 3 + \frac{9}{2} + \frac{9}{2}\right) \\
&= e^{-3} \times 13 \\
&= 0.647\,.
\end{aligned}
\tag{5.4.18}
$$

(b) **(i)** From Def. 2.2.5, we know that the conditional probability of A given B is,

$$
P(A|B) = \frac{P(A \cap B)}{P(B)}\,. \tag{5.4.19}
$$

Hence, we need to determine $P(A \cap B)$. Using Prop. 2.2.1,

$$
\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
0.9 &= 0.5 + 0.6 - P(A \cap B) \\
\Longrightarrow\ P(A \cap B) &= 0.5 + 0.6 - 0.9 \\
&= 0.2\,.
\end{aligned}
\tag{5.4.20}
$$

Now, we can substitute in the above to find

$$
\begin{aligned}
P(A|B) &= \frac{P(A \cap B)}{P(B)} \\
&= \frac{0.2}{0.6} \\
&= \frac{1}{3}\,.
\end{aligned}
\tag{5.4.21}
$$

(ii) See Fig. 5.2.

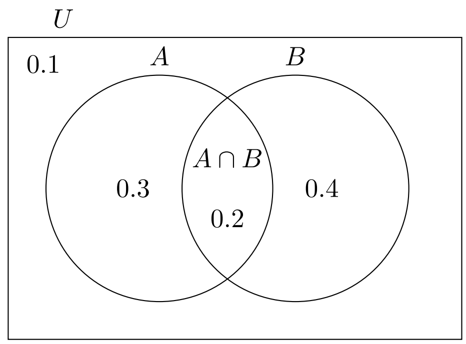


Figure 5.2: Venn Diagram.

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Chapter 6

2006

6.1 Module 1: Managing Uncertainty

1. **(a)** Given the proposition $\boldsymbol{x} \implies \boldsymbol{y}$,

   (i) $(\sim \boldsymbol{x} \implies \sim \boldsymbol{y})$ is the Inverse.

   (ii) $(\boldsymbol{y} \implies \boldsymbol{x})$ is the Converse.

   (iii) $(\sim \boldsymbol{y} \implies \sim \boldsymbol{x})$ is the Contrapositive.

   (b) **(i)** We can complete the truth table as follows

$\boldsymbol{x}$	$\boldsymbol{y}$	$\sim \boldsymbol{x}$	$\sim \boldsymbol{y}$	$\boldsymbol{x} \implies \boldsymbol{y}$	$\sim \boldsymbol{y} \implies \sim \boldsymbol{x}$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	0	0	1	1

Table 6.1: Truth Table of $\boldsymbol{x} \implies \boldsymbol{y}$ and $\sim \boldsymbol{y} \implies \sim \boldsymbol{x}$.

   (ii) From Table 6.1, we can see that the truth values of $\boldsymbol{x} \implies \boldsymbol{y}$ and $\sim \boldsymbol{y} \implies \sim \boldsymbol{x}$ are the same. Therefore, both expressions are logically equivalent.

   (c) **(i)** If $\boldsymbol{p}$ is the proposition "You have your cake" which is equivalent to "You do not eat your cake", then "You have your cake and eat it" can be expressed as,

$$\boldsymbol{p} \wedge \sim \boldsymbol{p}. \tag{6.1.1}$$

   (ii) The following truth can be constructed for the proposition $\boldsymbol{p} \wedge \sim \boldsymbol{p}$,

$\boldsymbol{p}$	$\sim \boldsymbol{p}$	$\boldsymbol{p} \wedge \sim \boldsymbol{p}$
0	1	0
1	0	0

Table 6.2: Truth Table of $\boldsymbol{p} \wedge \sim \boldsymbol{p}$.

From Table 6.2 and Def. 1.3.8, we can see that $\boldsymbol{p} \wedge \sim \boldsymbol{p}$ is a contradiction because the truth table is alwaays 0.

(d) We can construct the following activity network from the information given,

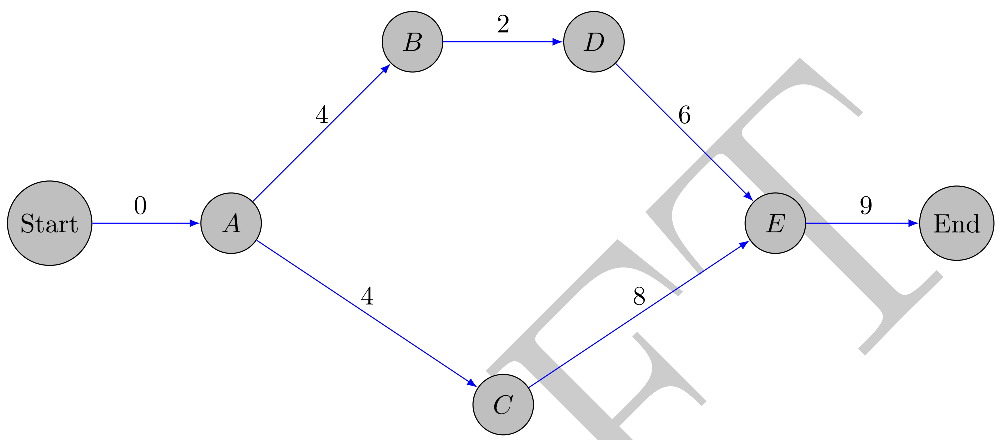

Figure 6.1: Activity Network of the project.

2. (a) Let us define x as "the number of large trucks used" and y as "the number of small trucks used". The objective function that we want to minimize is,

$$C = 240x + 100y\,. \tag{6.2.1}$$

The constraints in this linear programming model are as follows,

$$\begin{aligned} \text{Constraint of removed material: } & 12(5x + 2y) \geq 1200\,, \\ \text{Constraint of drivers: } & x + y \leq 50\,, \\ \text{Constraint of large trucks: } & x \leq 25\,, \\ \text{Constraint of small trucks: } & y \geq 15\,. \end{aligned} \tag{6.2.2}$$

(b) (i) See Fig. 6.2.

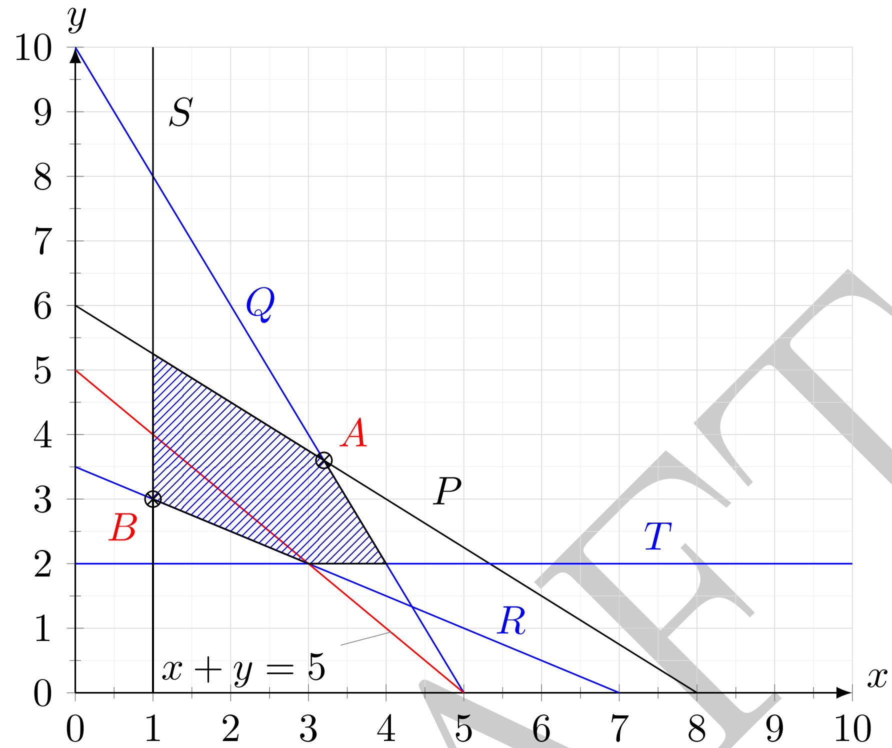

Figure 6.2: Linear Programming Graph.

(ii) The feasible region is shaded in the graph.

(iii) The point A, where $x + y$ is maximised, and the point B, where $x + y$ is minimised, are labeled on the graph as the relevant vertices of the feasible region.

6.2 Module 2: Probability and Distributions

3. (a) We are given the cumulative distribution function, $F(x)$, of a continuous random variable X.

(i) From Prop. 2.9.2, we know that $P(X = a)$ for any real a is 0. So,

$$P\left(X = \frac{1}{2}\right) = 0\,. \tag{6.3.1}$$

(ii) We can compute probabilities via the c.d.f using Note 2.9.3,

$$P\left(\frac{1}{2} < X \leq \frac{3}{4}\right) = F\left(\frac{3}{4}\right) - F\left(\frac{1}{2}\right)\,. \tag{6.3.2}$$

Substituting the given c.d.f yields,

$$
\begin{aligned}
P\left(\frac{1}{2} < X \leq \frac{3}{4}\right) &= \frac{\left(\frac{3}{4}\right)^2 + \frac{3}{4}}{2} - \frac{\left(\frac{1}{2}\right)^2 + \frac{1}{2}}{2} \\
&= \frac{\frac{9}{16} + \frac{3}{4}}{2} - \frac{\frac{1}{4} + \frac{1}{2}}{2} \\
&= \frac{\frac{21}{16}}{2} - \frac{\frac{3}{4}}{2} \\
&= \frac{\frac{21}{16} - \frac{3}{4}}{2} \\
&= \frac{\frac{9}{16}}{2} \\
&= \frac{9}{32}\,.
\end{aligned} \tag{6.3.3}
$$

(iii) We know that the c.d.f, F, and p.d.f, f, are related in Def. 2.9.4, via

$$
f(x) = \frac{\mathrm{d}}{\mathrm{d}x} F(x) \tag{6.3.4}
$$

Now, we must compute the p.d.f in the three regions over which F is defined. The only non-trivial calculation is when $0 \leq x < 1$,

$$
\begin{aligned}
f(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2 + x}{2}\right) \\
&= x + \frac{1}{2}\,.
\end{aligned} \tag{6.3.5}
$$

Thus, $f(x)$ is defined as,

$$
f(x) = \begin{cases} 0, & x < 0 \\ x + \frac{1}{2}, & 0 \leq x < 1 \\ 0, & x \geq 1 \end{cases} \tag{6.3.6}
$$

(b) The probability density function, $f(y)$, of the continuous random variable, Y, is given.

(i) We can use the property that probabilities must sum to one, Prop. 2.9.1 for continuous random variables,

$$
\int_{-\infty}^{\infty} f(y)\,\mathrm{d}y = 1\,. \tag{6.3.7}
$$

Performing this integral requires us to split the integral into three regions over which f is defined,

$$
\int_{-\infty}^{\infty} f(y)\,\mathrm{d}y = \int_{-\infty}^{1} f(y)\,\mathrm{d}y + \int_{1}^{2} f(y)\,\mathrm{d}y + \int_{2}^{\infty} f(y)\,\mathrm{d}y \tag{6.3.8}
$$

Since we know how the functions are defined in each region, we can solve for k,

$$
\begin{aligned}
0 + \int_1^2 \left(ky^3\right)\,\mathrm{d}y + 0 &= 1 \\
k \times \int_1^2 \left(y^3\right)\mathrm{d}y &= 1 \\
k \times \left[\frac{y^4}{4}\right]_1^2 &= 1 \\
k \times \left[\frac{2^4}{4} - \frac{1^4}{4}\right] &= 1 \\
k \times \left[\frac{16}{4} - \frac{1}{4}\right] &= 1 \\
\frac{15k}{4} &= 1 \\
\implies k &= \frac{4}{15}\,.
\end{aligned} \tag{6.3.9}
$$

(ii) From Def. 2.9.2, we know that the expectation of a continuous random variable Y is given by,

$$
E(Y) = \int_{-\infty}^{\infty} yf(y)\mathrm{d}y \tag{6.3.10}
$$

Splitting up the integral as before,

$$
E(Y) = \int_{-\infty}^{1} yf(y)\,\mathrm{d}y + \int_1^2 yf(y)\,\mathrm{d}y + \int_2^{\infty} yf(y)\,\mathrm{d}y\,, \tag{6.3.11}
$$

allows us to evaluate

$$
\begin{aligned}
E(Y) &= 0 + \int_1^2 \left(\frac{4y^4}{15}\right)\mathrm{d}y + 0 \\
&= \frac{4}{15} \times \left[\frac{y^5}{5}\right]_1^2 \\
&= \frac{4}{15} \times \left[\frac{2^5}{5} - \frac{1^5}{5}\right] \\
&= \frac{4}{15} \times \left[\frac{32}{5} - \frac{1}{5}\right] \\
&= \frac{124}{75}\,.
\end{aligned} \tag{6.3.12}
$$

(iii) We want to find the value of y_{30} such that,

$$
P(Y \leq y_{30}) = \frac{30}{100} \tag{6.3.13}
$$

From Note 2.9.1, we know that given the p.d.f., we can find the probability by integrating $f(y)$ in the appropriate region,

$$P(Y \leq y_{30}) = \int_{-\infty}^{y_{30}} f(y)\,\mathrm{d}y\,. \tag{6.3.14}$$

Since $f(y)$ is defined piecewise, we must split the integral up accordingly,

$$P(Y \leq y_{30}) = \int_{-\infty}^{1} f(y)\,\mathrm{d}y + \int_{1}^{y_{30}} f(y)\,\mathrm{d}y \tag{6.3.15}$$

Evaluating the integral over the different regions yields,

$$\begin{aligned} P(Y \leq y_{30}) &= 0 + \int_{1}^{y_{30}} f(y)\mathrm{d}y \\ &= \frac{4}{15} \times \left[\frac{y^4}{4}\right]_1^{y_{30}} \\ &= \frac{4}{15} \times \left[\frac{{y_{30}}^4}{4} - \frac{1^4}{4}\right] \\ &= \frac{1}{15} \times \left[{y_{30}}^4 - 1\right] \end{aligned} \tag{6.3.16}$$

Now, we can solve for y_{30} by recalling that $P(Y \leq y_{30}) = 0.3$,

$$\begin{aligned} \frac{1}{15} \times \left[{y_{30}}^4 - 1\right] &= 0.03 \\ {y_{30}}^4 - 1 &= 4.5 \\ \implies y_{30} &= \sqrt[4]{5.5} \\ &\approx 1.531\,. \end{aligned} \tag{6.3.17}$$

4. (a) (i) Using Def. 2.1.6, we can arrange the class in a straight line as follows,

$$\begin{aligned} \text{Straight line arrangements} = {}^{12}P_{12} &= \frac{12!}{(12-12)!} \\ &= \frac{12!}{0!} \\ &= \frac{12!}{1} \\ &= 12! \end{aligned} \tag{6.4.1}$$

(ii) We should first notice that every straight line arrangement of the class can be put around a circle. However, we have 12 different positions around the circle in which we can start the line. Because of this, exactly 12 straight line arrangements of the class will be equivalent when placed around the circle. Thus, we get that the number of circular arrangements of the class is,

$$\begin{aligned} \text{Circular arrangements} &= \frac{12!}{12} \\ &= 11! \end{aligned} \tag{6.4.2}$$

(iii) We will proceed by splitting the class into 6 groups to be arranged. 5 groups will contain 1 distinct male student and the 6th group will have the 7 female students. By Def. 2.1.1, we must arrange the group of 7 female students and multiply that number by the total arrangements of the 6 groups. This will give,

$$\begin{aligned}\text{Arrangemments w/ Females next to each other in a line} &= {}^{7}P_{7} \times {}^{6}P_{6} \\ &= \frac{7!}{(7-7)!} \times \frac{6!}{(6-6)!} \\ &= \frac{7!}{0!} \times \frac{6!}{0!} \\ &= 7! \times 6! \qquad (6.4.3)\end{aligned}$$

(iv) We will group the tallest male and female students as 1 group to consider. Thus, we will multiply the number of arrangements of the 11 groups (10 distinct students and 1 group with the tallest students) by the number of ways that the two tall students can be arranged. In order to arrange them in a straight line, we get that,

$$\begin{aligned}\text{Arrangements w/ tall students next to each other in a line} &= {}^{11}P_{11} \times^{2} P_{2} \\ &= \frac{11!}{(11-11)!} \times \frac{2!}{(2-2)!} \\ &= \frac{11!}{0!} \times \frac{2!}{0!} \\ &= 11! \times 2! \qquad (6.4.4)\end{aligned}$$

Since we are arranging the 11 groups (10 distinct students and 1 group with the tallest students) around a cirlce, using similar reasoning from **(4)(a)(ii)**, we can get that,

$$\begin{aligned}\text{Arrangements w/ tallest students next to each other in a circle} &= \frac{11! \times 2!}{11} \\ &= 10! \times 2! \qquad (6.4.5)\end{aligned}$$

(v) We will first calculate the number of arrangements with female students. The line can be formed as follows,

$$F_7 : \text{Arrangement of remaining 10 students} : F_6 . \qquad (6.4.6)$$

where F represents a female student and the subscripts 7 and 6 represent the number of choices of students for the end positions.
Thus, using Def. 2.1.1, we get that,

$$\begin{aligned}\text{Arrangements with 2 females on the ends} &= 7 \times^{10} P_{10} \times 6 \\ &= 7 \times \frac{10!}{(10-10)!} \times 6 \\ &= 42 \times \frac{10!}{(0)!} \\ &= 42 \times 10! \qquad (6.4.7)\end{aligned}$$

Similarly, the line with male students on the ends can be formed as,

$$M_5 : \text{Arrangement of remaining 10 students} : M_4\,. \tag{6.4.8}$$

where M represents a male student and the subscripts 5 and 4 represent the number of choices of students for the end positions.
Thus, using Def. 2.1.1, we get that,

$$\begin{aligned}\text{Arrangements with 2 males on the ends} &= 5 \times^{10} P_{10} \times 4 \\ &= 5 \times \frac{10!}{(10-10)!} \times 4 \\ &= 20 \times \frac{10!}{(0)!} \\ &= 20 \times 10! \end{aligned} \tag{6.4.9}$$

Thus, the total number of arrangements with students of the same sex on the ends is,

$$\begin{aligned}\text{Arrangements with 2 students of the same sex on the ends} &= 42 \times 10! + 20 \times 10! \\ &= 62 \times 10! \end{aligned} \tag{6.4.10}$$

(b) Let us first calculate the probability that a male is chosen first and then a female student as follows,

$$\begin{aligned} P(1^{st}M,\ 2^{nd}F) &= P(M\ 1^{st}) \times P(F\ 2^{nd}) \\ &= \frac{5}{12} \times \frac{7}{11} \\ &= \frac{35}{132}\,. \end{aligned} \tag{6.4.11}$$

Next, the probability that a female is chosen first and then a male student is,

$$\begin{aligned} P(1^{st}F,\ 2^{nd}M) &= P(F\ 1^{st}) \times P(M\ 2^{nd}) \\ &= \frac{7}{12} \times \frac{5}{11} \\ &= \frac{35}{132}\,. \end{aligned} \tag{6.4.12}$$

Thus, the probability that 2 students chosen are of the different sexes is,

$$\begin{aligned} P(\text{Different sexes}) &= P(1^{st}M \wedge 2^{nd}F) + P(1^{st}F,\ 2^{nd}M) \\ &= \frac{35}{132} + \frac{35}{132} \\ &= \frac{35}{66}\,. \end{aligned} \tag{6.4.13}$$

Chapter 7

2007

7.1 Module 1: Managing Uncertainty

1. (a) Let x be the number of newspaper advertisements and let y be the number of television advertisements. The objective function C, the sale of cars, that we want to maximize can be expressed as,

$$C = 2x + 5y\,. \tag{7.1.1}$$

The constraints of this linear programming model are as follows,

$$\begin{aligned} \text{Total budget: } & 1500x + 5000y \leq 5000\,, \\ \text{Newspaper ad budget: } & 1500x \leq 30000\,, \\ \text{TV ad budget: } & 5000y \geq 25000\,, \\ \text{Relative number of advertisements: } & x \leq 2y\,. \end{aligned} \tag{7.1.2}$$

(b) (i) The feasible region is shaded in Fig. 7.1.

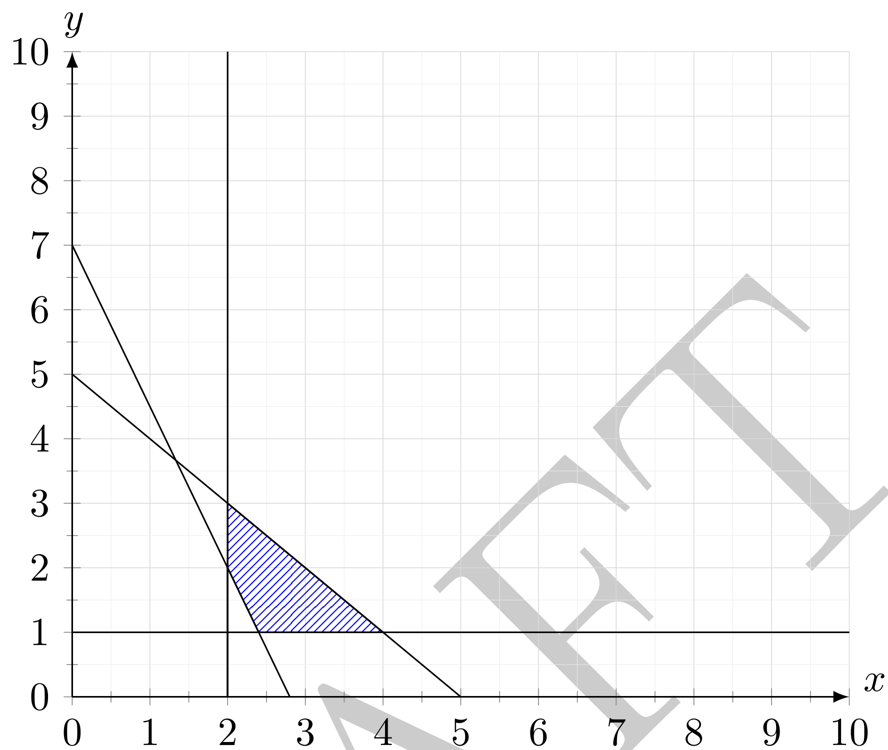

Figure 7.1: Linear Programming Graph.

(ii) We can solve the problem by performing a Tour of the Vertices (Def. **??**),

Vertice	$P = 2x + 7y$
(2, 2)	18
(2, 3)	25
(4, 1)	15
(2.4, 1)	11.8

Table 7.1: Tour of Vertices

Thus, the maximum value of P is 25, when $x = 2$ and $y = 3$.

2. (a) The Hungarian algorithm can be applied to the data given in order to minimise the total race time for the relay team consisting of four runners, W, X, Y and Z,

$$\begin{matrix} 48 & 46 & 50 & 44 \\ 49 & 45 & 46 & 49 \\ 47 & 46 & 48 & 44 \\ 51 & 48 & 47 & 45 \end{matrix}$$

Matrix From question

$$\begin{matrix} 4 & 2 & 6 & 0 \\ 4 & 0 & 1 & 4 \\ 3 & 2 & 4 & 0 \\ 6 & 3 & 2 & 0 \end{matrix}$$

Reducing Rows

$$\begin{matrix} 1 & 2 & 5 & 0 \\ 1 & 0 & 0 & 4 \\ 0 & 2 & 3 & 0 \\ 3 & 3 & 1 & 0 \end{matrix}$$

Reducing Columns

$$\begin{matrix} 1 & 2 & 5 & 0 \\ 1 & 0 & 0 & 4 \\ 0 & 2 & 3 & 0 \\ 3 & 3 & 1 & 0 \end{matrix}$$

Shading 0's

$$\begin{matrix} 0 & 1 & 4 & 0 \\ 1 & 0 & 0 & 6 \\ 0 & 2 & 3 & 2 \\ 2 & 2 & 0 & 0 \end{matrix}$$

Applying Step 5, 1.1.2

$$\begin{matrix} 0 & 1 & 4 & 0 \\ 1 & 0 & 0 & 6 \\ 0 & 2 & 3 & 2 \\ 2 & 2 & 0 & 0 \end{matrix}$$

Shading 0's

Table 7.2: Showing the steps of the Hungarian Algorithm.

From Table 7.2, the possible pairings of the runners and the parts of the race are,

$$\begin{aligned} W &\to 1 \text{ or } 4\,, \\ X &\to 2 \text{ or } 3\,, \\ Y &\to 1\,, \\ Z &\to 3 \text{ or } 4\,. \end{aligned} \tag{7.2.1}$$

Thus, the optimal pairing to minimize the total race time is,

$$\begin{aligned} W &\to 4\,, \\ X &\to 2\,, \\ Y &\to 1\,, \\ Z &\to 3\,. \end{aligned} \tag{7.2.2}$$

(b) We can construct the following activity network to represent the given data,

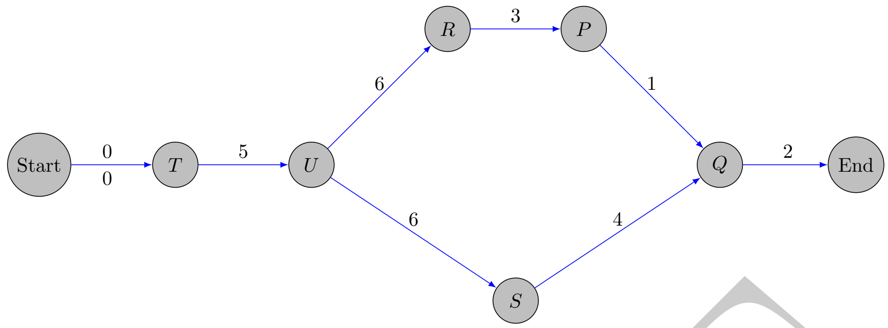

Figure 7.2: Activity Network of the project.

(c) **(i)** We can calculate the earliest and latest start times for each activity in the network as follows:

Activity	Earliest Start Time	Latest Start Time	Float Time
A	0	0	0
B	6	7	1
C	6	6	0
D	15	15	0
E	21	22	1
F	21	21	0
G	34	34	0

Table 7.3: Float times of the activities.

(ii) **a)** Using the float times from Table 7.3, we can identify the critical path as the path that has zero float time (Def. 1.2.7).

$$\text{Start} \to A \to C \to D \to F \to G \to \text{Finish}\,. \tag{7.2.3}$$

b) The latest finish time of E can be calucalated by the sum of the latest start time of E and the duration of E. Therefore, from Table 7.3, the latest finish time of E is $22 + 12 = 34$.

7.2 Module 2: Probability and Distributions

3. **(a)** Let X be the discrete random variable that represents "the number of 3's thrown when a dice is rolled 4 times."

$$X \sim \text{Bin}\left(4, \frac{1}{6}\right)\,. \tag{7.3.1}$$

From Def. 2.5.1, the probabilities of a Binomial distribution are,

$$P(X = x) = \binom{4}{x} \times \left(\frac{1}{6}\right)^{x} \times \left(1 - \frac{1}{6}\right)^{4-x}\,. \tag{7.3.2}$$

Using Eq. (7.3.2), we can then calculate the expected probabilities, assuming a Binomial distribution,

$$\begin{aligned}
P(X=0) &= \binom{4}{0} \times \left(\frac{1}{6}\right)^{0} \times \left(1-\frac{1}{6}\right)^{4-0} \\
&= 1 \times 1 \times \left(\frac{5}{6}\right)^{4} \\
&= \frac{625}{1296}\,,
\end{aligned} \tag{7.3.3}$$

$$\begin{aligned}
P(X=1) &= \binom{4}{1} \times \left(\frac{1}{6}\right)^{1} \times \left(1-\frac{1}{6}\right)^{4-1} \\
&= 4 \times \frac{1}{6} \times \left(\frac{5}{6}\right)^{3} \\
&= \frac{500}{1296}\,,
\end{aligned} \tag{7.3.4}$$

$$\begin{aligned}
P(X=2) &= \binom{4}{2} \times \left(\frac{1}{6}\right)^{2} \times \left(1-\frac{1}{6}\right)^{4-2} \\
&= 6 \times \left(\frac{1}{6}\right)^{2} \times \left(\frac{5}{6}\right)^{2} \\
&= \frac{150}{1296}\,,
\end{aligned} \tag{7.3.5}$$

$$\begin{aligned}
P(X=3) &= \binom{4}{3} \times \left(\frac{1}{6}\right)^{3} \times \left(1-\frac{1}{6}\right)^{4-3} \\
&= 4 \times \left(\frac{1}{6}\right)^{3} \times \left(\frac{5}{6}\right) \\
&= \frac{20}{1296}\,,
\end{aligned} \tag{7.3.6}$$

and,

$$\begin{aligned}
P(X=4) &= \binom{4}{4} \times \left(\frac{1}{6}\right)^{4} \times \left(1-\frac{1}{6}\right)^{4-4} \\
&= 1 \times \left(\frac{1}{6}\right)^{4} \times 1 \\
&= \frac{1}{1296}\,.
\end{aligned} \tag{7.3.7}$$

Now that we have our expected probailities, we can compute the expected frequencies. The results are summarized in Table 7.4.

$X = x$	$P(X = x)$	Expected frequency, Total $\times P(X = x)$
0	0.482	48.2
1	0.386	38.6
2	0.116	11.6
3	0.015	1.5
4	0.001	0.1

Table 7.4: Expected Frequencies.

(b) We are asked to perform a χ^2 goodness-of-fit test at the 5% significance level. From Section(CHI Squared notes), we know that,

$$\chi^2_{\text{calc.}} = \sum_{\forall k} \left(\frac{(O_k - E_k)^2}{E_k} \right) . \tag{7.3.8}$$

Our hypotheses are,

- H_0: The data follows a binomial distribution with parameters $n = 4$ and $p = \frac{1}{6}$.
- H_1: The data does not follow a binomial distribution with parameters $n = 4$ and $p = \frac{1}{6}$.

We should note that the expected frequencies for $X = 3$ and $X = 4$ are both less than 5. Therefore, we must combine both classes with $X = 2$ as follows,

$X = x$	Observed Frequency, O	Expected Frequency, E	$\frac{(O-E)^2}{E}$
0	57	48.2	1.607
1	30	38.6	1.916
≥ 2	13	13.2	0.003
Total, $\chi^2_{\text{calc.}}$			3.526

Table 7.5: Expected and Observed Frequencies of experiment.

From Table 7.5, we see that our number of degrees of freedom, ν, is,

$$\nu = 3 - 1 = 2 . \tag{7.3.9}$$

We reject H_0 if and only if,

$$\begin{aligned} & \chi^2_{\text{calc.}} > \chi^2_{\alpha}(\nu) \\ \implies & \chi^2_{\text{calc.}} > \chi^2_{0.05}(2) \\ \implies & \chi^2_{\text{calc.}} > 5.991 . \end{aligned} \tag{7.3.10}$$

Since our value of $\chi^2_{\text{calc.}} = 3.526$ is not greater than $\chi^2_{0.05}(2) = 5.991$, we fail to reject H_0.

4. (a) The probability density function, $f(x)$, of a continuous random variable X is given.

We can use the property that probabilities must sum to one, Prop. 2.9.1 for continuous random variables,

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1\,. \tag{7.4.1}$$

Since $f(x)$ is only defined for $1 \leq x \leq 5$, the integral reduces to

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x &= \int_{-\infty}^{1} f(x)\,\mathrm{d}x + \int_{1}^{5} f(x)\,\mathrm{d}x + \int_{5}^{\infty} f(x)\,\mathrm{d}x \\ &= 0 + \int_{1}^{5} (ax)\,\mathrm{d}x + 0\,. \end{aligned} \tag{7.4.2}$$

Evaluating and equating to 1, we can solve for a,

$$\begin{aligned} a \times \left[\frac{x^2}{2}\right]_1^5 &= 1 \\ a \times \left(\frac{5^2}{2} - \frac{1^2}{2}\right) &= 1 \\ a \times \frac{25-1}{2} &= 1 \\ 12a &= 1 \\ \implies a &= \frac{1}{12}\,. \end{aligned} \tag{7.4.3}$$

(b) From Note 2.9.1, we know that we must integrate the p.d.f over the appropriate region,

$$\begin{aligned} P(X < 3) &= P(-\infty < X < 3) \\ &= \int_{-\infty}^{3} f(x)\,\mathrm{d}x\,. \end{aligned} \tag{7.4.4}$$

Again, splitting up the integral in the appropriate intervals,

$$P(X < 3) = \int_{-\infty}^{1} f(x)\,\mathrm{d}x + \int_{1}^{3} f(x)\,\mathrm{d}x\,, \tag{7.4.5}$$

allows us to substitute and evaluate as,

$$\begin{aligned} P(X<3) &= 0 + \int_1^3 \frac{x}{12}\,\mathrm{d}x \\ &= \frac{1}{12} \times \left[\frac{x^2}{2}\right]_1^3 \\ &= \frac{1}{12} \times \left(\frac{3^2}{2} - \frac{1^2}{2}\right) \\ &= \frac{1}{12} \times \left(\frac{9}{2} - \frac{1}{2}\right) \\ &= \frac{1}{12} \times \frac{8}{2} \\ &= \frac{1}{3}\,. \end{aligned} \tag{7.4.6}$$

(c) Using Def. 2.9.2, we know that $E(X)$ is given by,

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x\,. \tag{7.4.7}$$

Splitting up the integral over the appropriate intervals,

$$E(X) = \int_{-\infty}^{1} x f(x)\,\mathrm{d}x + \int_1^5 x f(x)\,\mathrm{d}x + \int_5^{\infty} x f(x)\,\mathrm{d}x\,, \tag{7.4.8}$$

allows us to substitute and evaluate,

$$\begin{aligned} E(X) &= 0 + \int_1^5 x\left(\frac{x}{12}\right)\,\mathrm{d}x + 0 \\ &= \frac{1}{12} \times \int_1^5 x^2\,\mathrm{d}x \\ &= \frac{1}{12} \times \left[\frac{x^3}{3}\right]_1^5 \\ &= \frac{1}{12} \times \left(\frac{5^3}{3} - \frac{1^3}{3}\right) \\ &= \frac{1}{12} \times \frac{124}{3} \\ &= \frac{124}{36}\,. \end{aligned} \tag{7.4.9}$$

To compute $\mathrm{Var}(X)$, we use Def. 2.9.3,

$$\mathrm{Var}(X) = E(X^2) - E(X)^2\,. \tag{7.4.10}$$

Using Eq. (2.9.9), we can calculate $E(X^2)$,

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x\,. \tag{7.4.11}$$

As before, we split up the integral,

$$E(X^2) = \int_{-\infty}^{1} x^2 f(x)\,\mathrm{d}x + \int_{1}^{5} x^2 f(x)\,\mathrm{d}x + \int_{5}^{\infty} x^2 f(x)\,\mathrm{d}x\,, \tag{7.4.12}$$

substitute, and evaluate,

$$\begin{aligned} E(X^2) &= 0 + \int_1^5 x^2 \left(\frac{x}{12}\right) \mathrm{d}x + 0 \\ &= \frac{1}{12} \times \int_1^5 x^3 \mathrm{d}x \\ &= \frac{1}{12} \times \left[\frac{x^4}{4}\right]_1^5 \\ &= \frac{1}{12} \times \left(\frac{5^4}{4} - \frac{1^4}{4}\right) \\ &= \frac{1}{12} \times \frac{624}{4} \\ &= \frac{624}{48}\,. \end{aligned} \tag{7.4.13}$$

Then, substituting these values into Eq. (7.4.10), we find,

$$\begin{aligned} \mathrm{Var}(X) &= E(X^2) - E(X)^2 \\ &= \frac{624}{48} - \left(\frac{124}{36}\right)^2 \\ &= \frac{624}{48} - \frac{961}{81} \\ &= \frac{92}{81}\,. \end{aligned} \tag{7.4.14}$$

(d) From Def. 2.9.4, the cumulative distribution function is given by,

$$F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t\,. \tag{7.4.15}$$

We can consider this for three possible values of x.[1]
For $x < 1$,

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)\,\mathrm{d}t \\ &= 0\,. \end{aligned} \tag{7.4.16}$$

[1] Technically the domain was already restricted to $1 \leq x \leq 5$, so there is only one region. We decided to stick with $x \in \mathrm{R}$ for consistency. But the answer does not change with the way we have presented it.

For $1 \leq x < 5$,

$$
\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(t)\,\mathrm{d}t \\
&= \int_{-\infty}^{1} f(t)\,\mathrm{d}t + \int_{1}^{x} f(t)\,\mathrm{d}t \\
&= 0 + \int_{1}^{x} \left(\frac{t}{12}\right)\,\mathrm{d}t \\
&= \frac{1}{12} \times \left[\frac{t^2}{2}\right]_1^a \\
&= \frac{1}{12} \times \left(\frac{x^2}{2} - \frac{1}{2}\right) \\
&= \frac{x^2}{24} - \frac{1}{24}\,. \qquad (7.4.17)
\end{aligned}
$$

For $x > 5$,

$$
\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(t)\,\mathrm{d}t \\
&= \int_{-\infty}^{1} f(t)\,\mathrm{d}t + \int_{1}^{5} f(t)\,\mathrm{d}t + \int_{5}^{x} f(t)\,\mathrm{d}t \\
&= 0 + 1 + 0\,. \qquad (7.4.18)
\end{aligned}
$$

Therefore, the cumulative distribution function is given as,

$$
F(x) = \begin{cases} 0, & x < 1 \\ \frac{x^2}{24} - \frac{1}{24}, & 1 \leq x < 5 \\ 1, & x \geq 5 \end{cases} \qquad (7.4.19)
$$

(e) Recall from Prop. 2.9.3, that the median M, is the value of $x = M$ such that

$$
F(M) = \frac{1}{2}\,. \qquad (7.4.20)
$$

Since $0 \leq F(M) \leq 1$, by inspecting our definition of F above, we know $1 \leq M < 5$. Thus, we can equate and solve as follows,

$$
\begin{aligned}
\frac{M^2}{24} - \frac{1}{24} &= \frac{1}{2} \\
M^2 - 1 &= 12 \\
M^2 &= 13 \\
\implies M &= \sqrt{13}\,. \qquad (7.4.21)
\end{aligned}
$$

(f) From Section 2.12, we know that,

$$
E(3X + 2) = 3\,E(X) + 2\,. \qquad (7.4.22)
$$

Substituting for $E(X)$, we have

$$E(3X+2) = 3 \times \frac{124}{36} + 2 = \frac{37}{3}\,. \tag{7.4.23}$$

Similarly, we also know that

$$\mathrm{Var}(3X+2) = 3^2\,\mathrm{Var}(X)\,. \tag{7.4.24}$$

Substituting for $\mathrm{Var}(X)$, we have

$$\mathrm{Var}(3X+2) = 9 \times \frac{92}{81} = \frac{92}{9}\,. \tag{7.4.25}$$

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Chapter 8

2008

8.1 Module 1: Managing Uncertainty

1. **(a)** The objective function P, profit, that we want to maximize is,

$$P = 15x + 27y\,. \tag{8.1.1}$$

Using x as the number of units of product X and y as the number of units of product Y, our inequalities are,

$$\begin{aligned}
\text{Minimum constraint: } & x \geq 0\,,\\
\text{Minimum constraint: } & y \geq 0\,,\\
\text{Time constraint: } & 6x + 9y \leq 360 \implies 2x + 3y \leq 120\,,\\
\text{Cost constraint: } & 15x + 9y \leq 675 \implies 5x + 3y \leq 225\,.
\end{aligned} \tag{8.1.2}$$

(b) The feasible region of this linear programming problem is shaded in Fig. 8.1.

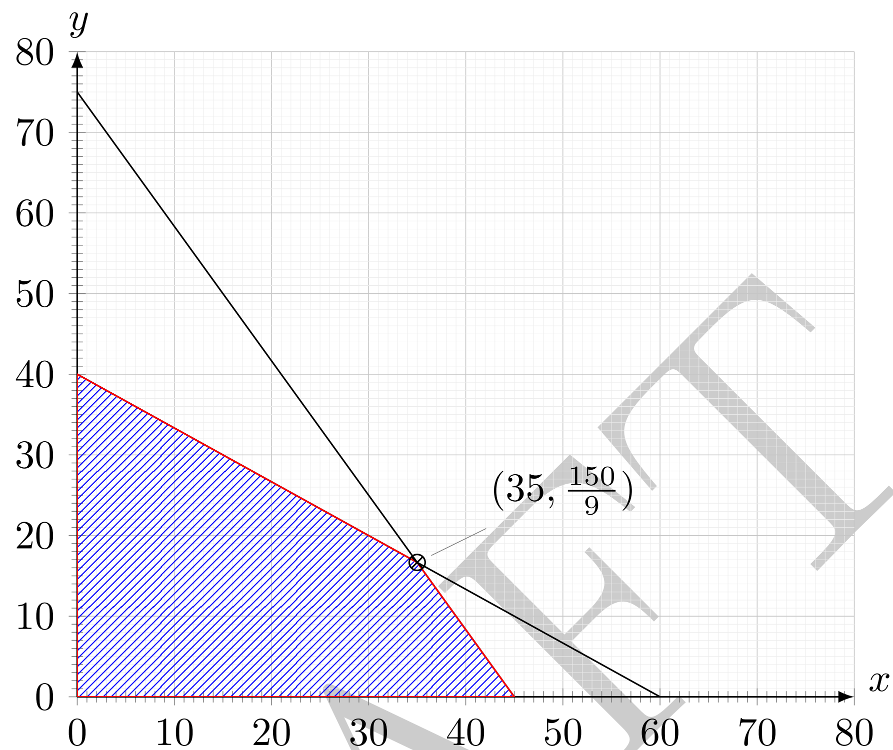


Figure 8.1: Linear Programming Graph.

(c) We can perform a tour of the vertices (Def. **??**) to find the values of x and y that maximize the objective function P

Vertice	$P = 15x + 27y$
(0, 0)	0
(0, 40)	1080
(35, $\frac{150}{9}$)	705
(45, 0)	675

Table 8.1: Tour of Vertices

Thus, from Table 8.1, the maximum value of P is 1080, when $x = 0$ and $y = 40$.

2. **(a)** The proposition $(\sim \boldsymbol{p} \wedge \sim \boldsymbol{q})$ can be expressed in words as "He is not tall and he is not happy."

(b) Let $\boldsymbol{p}$ = "London is in England," and let $\boldsymbol{q}$ = "$2 \times 3 = 5$".

"London is in England or 2 x 3 = 5" can be expressed as the proposition $\boldsymbol{p} \vee \boldsymbol{q}$.

We know that $\boldsymbol{p}$ has a truth value of 1 and $\boldsymbol{q}$ has a truth value of 0.

Thus, by referring to Def. 1.3 we get that the truth value of the statement is equal to,

$$\begin{aligned} \boldsymbol{p} \vee \boldsymbol{q} &= \mathbf{1} \vee \mathbf{0} \\ &= \mathbf{1}\,. \end{aligned} \tag{8.2.1}$$

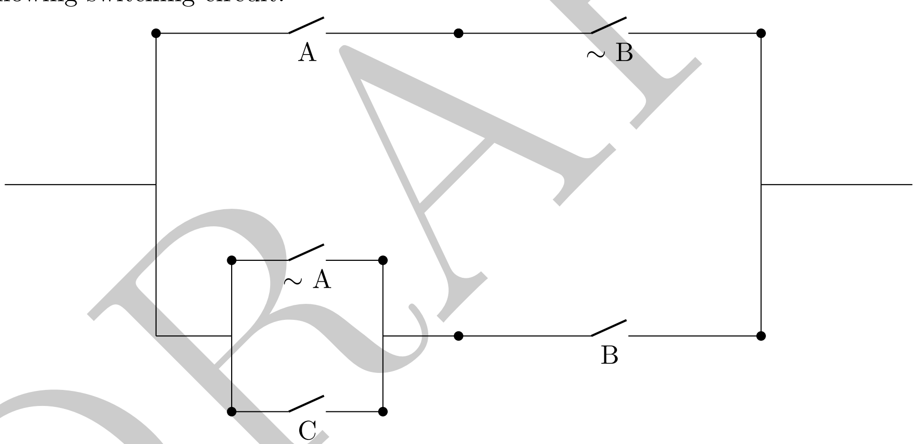
(c) We can construct the following truth table for the proposition $(\boldsymbol{p} \wedge \boldsymbol{q}) \implies (\boldsymbol{p} \vee \boldsymbol{q})$

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p} \wedge \boldsymbol{q}$	$\boldsymbol{p} \vee \boldsymbol{q}$	$(\boldsymbol{p} \wedge \boldsymbol{q}) \implies (\boldsymbol{p} \vee \boldsymbol{q})$
0	0	0	0	1
0	1	0	1	1
1	0	0	1	1
1	1	1	1	1

Table 8.2: Showing the truth values of $(\boldsymbol{p} \wedge \boldsymbol{q}) \implies (\boldsymbol{p} \vee \boldsymbol{q})$.

We can see that the proposition $(\boldsymbol{p} \wedge \boldsymbol{q}) \implies (\boldsymbol{p} \vee \boldsymbol{q})$ is a tautology (Def. 1.3.7) since it always has a truth value of 1.

(d) **(i)** We can represent the boolean expression $(\boldsymbol{A} \wedge \sim \boldsymbol{B}) \vee ((\sim \boldsymbol{A} \vee \boldsymbol{C}) \wedge \boldsymbol{B})$ as the following switching circuit.



(ii) We can represent the Boolean expression $\sim \boldsymbol{a} \wedge (\boldsymbol{a} \vee \boldsymbol{b})$ as the following logic circuit.

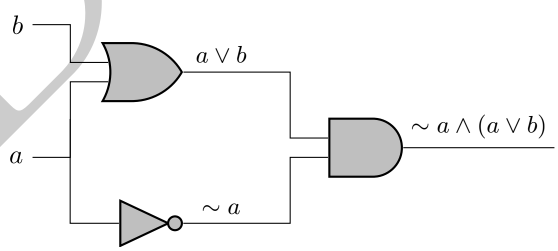


Figure 8.2: Showing the logic circuit of $\sim \boldsymbol{a} \wedge (\boldsymbol{a} \vee \boldsymbol{b})$.

(e) Let $\boldsymbol{Q} = (\boldsymbol{A} \wedge \boldsymbol{B}) \vee (\boldsymbol{A} \wedge \sim \boldsymbol{B}) \vee (\sim \boldsymbol{A} \wedge \sim \boldsymbol{B})$.

Let $\boldsymbol{R} = (\boldsymbol{A} \wedge \boldsymbol{B}) \vee (\boldsymbol{A} \wedge \sim \boldsymbol{B})$

By using the Distributive Law (Def. 1.3.9) in reverse and the Complement Law (Def. 1.3.4), $\boldsymbol{R}$ becomes

$$\begin{aligned} \boldsymbol{R} &\equiv \boldsymbol{A} \wedge (\boldsymbol{B} \vee \sim \boldsymbol{B}) \\ &\equiv \boldsymbol{A} \wedge \boldsymbol{T}\,. \end{aligned} \tag{8.2.2}$$

Therefore, we can rewrite $\boldsymbol{Q}$

$$\boldsymbol{Q} \equiv (\boldsymbol{A} \wedge \boldsymbol{T}) \vee (\sim \boldsymbol{A} \wedge \sim \boldsymbol{B})\,. \tag{8.2.3}$$

By using the Identity Law (Def. 1.3.2) and Distributive Law (Def. 1.3.9), $\boldsymbol{Q}$ becomes

$$\begin{aligned} \boldsymbol{Q} &\equiv \boldsymbol{A} \vee (\sim \boldsymbol{A} \wedge \sim \boldsymbol{B}) \\ &\equiv (\boldsymbol{A} \vee \sim \boldsymbol{A}) \wedge (\boldsymbol{A} \vee \sim \boldsymbol{B})\,. \end{aligned} \tag{8.2.4}$$

Finally by using the Complement (Def. 1.3.4) and Identity (Def. 1.3.2) Laws,

$$\begin{aligned} \boldsymbol{Q} &\equiv \boldsymbol{T} \wedge (\boldsymbol{A} \vee \sim \boldsymbol{B}) \\ &\equiv \boldsymbol{A} \vee \sim \boldsymbol{B}\,. \end{aligned} \tag{8.2.5}$$

Thus,

$$(\boldsymbol{A} \wedge \boldsymbol{B}) \vee (\boldsymbol{A} \wedge \sim \boldsymbol{B}) \vee (\sim \boldsymbol{A} \wedge \sim \boldsymbol{B}) \equiv (\boldsymbol{A} \vee \sim \boldsymbol{B})\,. \tag{8.2.6}$$

8.2 Module 2: Probability and Distributions

3. (a) We being by observing that the first digit must be greater than or equal to 5 in order for our number to be greater than 500,000. Thus, the first digit must be in the set {5, 6, 7, 8, 9}. Similarly, for our number to be odd, the final digit must be odd (from the set {5,7,9}). Since repetition is allowed when choosing our digits, we find that we have 5 choices for the first digit, 6 choices for the second, third, fourth and fifth digit and, 3 choices for our last digit. Thus, by Def. 2.1.1, we get that,

$$\begin{aligned} \text{Number of odd numbers greater than 500,000} &= 5 \times 6 \times 6 \times 6 \times 6 \times 3 \\ &= 19440\,. \end{aligned} \tag{8.3.1}$$

(b) (i) Given that there are no restrictions, we need to choose 4 people from a group of 13. From Def. 2.1.4, we get that,

$$\begin{aligned} {}^{13}C_4 &= \frac{13!}{4! \times (13-4)!} \\ &= \frac{13!}{4! \times 9!} \\ &= 715\,. \end{aligned} \tag{8.3.2}$$

(ii) If the team contains 2 girls, it must also contain 2 boys. Therefore, we must choose 2 girls and 2 boys from the group as follows,

$$\begin{aligned}\text{Team with exactly 2 girls} &= {}^6C_2 \times {}^7C_2 \\ &= \frac{6!}{2! \times (6-2)!} \times \frac{7!}{2! \times (7-2)!} \\ &= \frac{6!}{2! \times 4!} \times \frac{7!}{2! \times 5!} \\ &= 15 \times 21 \\ &= 315\,. \end{aligned} \tag{8.3.3}$$

(iii) In order for the team to have more girls than boys, there must either be 4 girls chosen or 3 girls chosen and 1 boy. Using Def. 2.1.2 and Def. 2.1.1, we see that this can be given as,

$$\begin{aligned}\text{Teams with more girls} &= \text{Teams with 4 girls} + \text{Teams with 3 girls, 1 boy} \\ &= {}^6C_4 + \left({}^6C_3 \times {}^7C_1\right) \\ &= \frac{6!}{4! \times (6-4)!} + \left(\frac{6!}{3! \times (6-3)!} \times \frac{7!}{1! \times (7-1)!}\right) \\ &= \frac{6!}{4! \times 2!} + \left(\frac{6!}{3! \times 3!} \times \frac{7!}{1! \times 6!}\right) \\ &= 15 + (20 \times 7) \\ &= 155\,. \end{aligned} \tag{8.3.4}$$

(c) **(i)** We are given that A and B are independent. From Def. 2.2.4, we know,

$$P(A \wedge B) = P(A) \times P(B)\,. \tag{8.3.5}$$

Using Def. 2.2.5, we can express $P(A|B)$ as,

$$P(A|B) = \frac{P(A \wedge B)}{P(B)}\,. \tag{8.3.6}$$

Combining this with the definition of independence, we see that,

$$\begin{aligned} P(A|B) &= \frac{P(A \wedge B)}{P(B)} \\ &= \frac{P(A) \times P(B)}{P(B)} \\ &= P(A)\,. \end{aligned} \tag{8.3.7}$$

Thus,

$$\begin{aligned} P(A) &= P(A|B) \\ &= \frac{2}{3}\,. \end{aligned} \tag{8.3.8}$$

(ii) Using $P(A) = \frac{2}{3}$, we can find $P(B)$ by using the independence,

$$P(A \wedge B) = P(A) \times P(B)\,. \tag{8.3.9}$$

Substituting the values

$$\begin{aligned} \frac{1}{3} &= \frac{2}{3} \times P(B) \\ \implies P(B) &= \frac{1}{3} \times \frac{3}{2} \\ &= \frac{1}{2}\,. \end{aligned} \tag{8.3.10}$$

(iii) We should note that,

$$P(A') = 1 - P(A)\,. \tag{8.3.11}$$

Using this, we can get that,

$$\begin{aligned} P(A' \wedge B') &= P(A') \times P(B') \\ &= (1 - P(A)) \times (1 - P(B)) \\ &= \left(1 - \frac{2}{3}\right) \times \left(1 - \frac{1}{2}\right) \\ &= \frac{1}{3} \times \frac{1}{2} \\ &= \frac{1}{6}\,. \end{aligned} \tag{8.3.12}$$

(d) From Def. 2.2.3, we know that A and B are mutually exclusive if and only if,

$$P(A \wedge B) = 0\,. \tag{8.3.13}$$

Given that we know $P(A \wedge B) = \frac{1}{3}$, we can conclude that A and B are not mutually exclusive.

4. (a) The probability density function, $P(X = x)$, of a discrete random variable X is given.

(i) We know, from Prop. 2.3.1, that the probabilities must sum to one,

$$\sum_{\forall i} P(X = x_i) = 1\,. \tag{8.4.1}$$

Since $P(X = x) \neq 0$ for $x \in \{1, 2, 3, 4\}$, we have,

$$\begin{aligned} P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) &= 1 \\ k + 2k + 3k + 4k &= 1 \\ 10k &= 1 \\ \implies k &= \frac{1}{10}\,. \end{aligned} \tag{8.4.2}$$

(ii) We can compute $E(X)$ for a discrete random variable, using Def. 2.3.2,

$$E(X) = \sum_{\forall i} x_i P(X = x_i)\,. \tag{8.4.3}$$

Evaluating, we find,

$$\begin{aligned} E(X) &= \left(1 \times \frac{1}{10}\right) + \left(2 \times \frac{2}{10}\right) + \left(3 \times \frac{3}{10}\right) + \left(4 \times \frac{4}{10}\right) \\ &= \frac{1}{10} + \frac{2}{5} + \frac{9}{10} + \frac{8}{5} \\ &= 3\,. \end{aligned} \tag{8.4.4}$$

(iii) Let the discrete random variable $Y = X_1 + X_2$, where X_1 and X_2 represent the discrete random variables for two independent observations of X.
To get $Y = 4$, we must consider the all the possibilities such that $X_1 = x_1$ and $X_2 = x_2$ and $x_1 + x_2 = 4$. So the probability is given by,

$$\begin{aligned} P(Y = 4) &= \sum_{\substack{1 \le x_1 \le 4 \\ 1 \le x_2 \le 4 \\ x_1 + x_2 = 4}} P(X_1 = x_1 \cap X_2 = x_2) \\ &= \sum_{1 \le x_1 \le 3} P(X_1 = x_1 \cap X_2 = 4 - x_1)\,, \end{aligned} \tag{8.4.5}$$

where we got the last line by using one of the constraints in the sum.
Further, since the events X_1 and X_2 are independent, from Def. 2.2.9, we know,

$$P(X_1 \cap X_2) = P(X_1) \times P(X_2)\,, \tag{8.4.6}$$

and so,

$$P(Y = 4) = \sum_{1 \le x_1 \le 4} P(X_1 = x_1) \times P(X_2 = 4 - x_1)\,. \tag{8.4.7}$$

Now, we can enumerate the possibilities and find the probability,

$$\begin{aligned} P(Y = 4) &= P(X_1 = 1 \cap X_2 = 3) + P(X_1 = 2 \cap X_2 = 2) + P(X_1 = 3 \cap X_2 = 1) \\ &= \left(\frac{1}{10} \times \frac{3}{10}\right) + \left(\frac{2}{10} \times \frac{2}{10}\right) + \left(\frac{3}{10} \times \frac{1}{10}\right) \\ &= \frac{1}{10}\,. \end{aligned} \tag{8.4.8}$$

(iv) Now, for general $Y = y$, we can simply adapt our formula above,

$$P(Y = y) = \sum_{1 \le x_1 \le 4} P(X_1 = x_1) \times P(X_2 = y - x_1)\,. \tag{8.4.9}$$

The calculations can be organized in a table, as in Table 8.3, and then summarized in Table 8.4.

Sum ($x_1 + x_2$)	$x_2 = 1$	$x_2 = 2$	$x_2 = 3$	$x_2 = 4$
$x_1 = 1$	$2\left(\frac{1}{10} \times \frac{1}{10} = \frac{1}{100}\right)$	$3\left(\frac{1}{10} \times \frac{2}{10} = \frac{2}{100}\right)$	$4\left(\frac{1}{10} \times \frac{3}{10} = \frac{3}{100}\right)$	$5\left(\frac{1}{10} \times \frac{4}{10} = \frac{4}{100}\right)$
$x_1 = 2$	$3\left(\frac{2}{10} \times \frac{1}{10} = \frac{2}{100}\right)$	$4\left(\frac{2}{10} \times \frac{2}{10} = \frac{4}{100}\right)$	$5\left(\frac{2}{10} \times \frac{3}{10} = \frac{6}{100}\right)$	$6\left(\frac{2}{10} \times \frac{4}{10} = \frac{8}{100}\right)$
$x_1 = 3$	$4\left(\frac{3}{10} \times \frac{1}{10} = \frac{3}{100}\right)$	$5\left(\frac{3}{10} \times \frac{2}{10} = \frac{6}{100}\right)$	$6\left(\frac{3}{10} \times \frac{3}{10} = \frac{9}{100}\right)$	$7\left(\frac{3}{10} \times \frac{4}{10} = \frac{12}{100}\right)$
$x_1 = 4$	$5\left(\frac{4}{10} \times \frac{1}{10} = \frac{4}{100}\right)$	$6\left(\frac{4}{10} \times \frac{2}{10} = \frac{8}{100}\right)$	$7\left(\frac{4}{10} \times \frac{3}{10} = \frac{12}{100}\right)$	$8\left(\frac{4}{10} \times \frac{4}{10} = \frac{16}{100}\right)$

Table 8.3: Sum and Probabilities of two independent observations of X

$P(Y = y)$	$P(Y = 2)$	$P(Y = 3)$	$P(Y = 4)$	$P(Y = 5)$	$P(Y = 6)$	$P(Y = 7)$	$P(Y = 8)$
	$\frac{1}{100}$	$\frac{4}{100}$	$\frac{10}{100}$	$\frac{20}{100}$	$\frac{25}{100}$	$\frac{24}{100}$	$\frac{16}{100}$

Table 8.4: Probabilities of $Y = y$

(v) From Def. 2.3.2, we know that

$$E(Y) = \sum_{\forall i} y_i P(Y = y_i)\,. \tag{8.4.10}$$

Substituting the values in Table 8.3, we get that,

$$\begin{aligned} E(Y) &= \left(2 \times \frac{1}{100}\right) + \left(3 \times \frac{4}{100}\right) + \ldots + \left(7 \times \frac{24}{100}\right) + \left(8 \times \frac{16}{100}\right) \\ &= \frac{2}{100} + \frac{12}{100} + \frac{40}{100} + \frac{100}{100} + \frac{150}{100} + \frac{168}{100} + \frac{128}{100} \\ &= 6\,. \end{aligned} \tag{8.4.11}$$

Next, we determine $\text{Var}(Y)$. From Def. 2.3.4, we know that,

$$\text{Var}(X) = E(Y^2) - (E(Y))^2\,. \tag{8.4.12}$$

Thus, we need to calculate $E(Y^2)$. From Def. 2.3.3, we have that,

$$E(Y^2) = \sum_{\forall i} y_i^2 P(Y = y_i)\,. \tag{8.4.13}$$

Substituting the values in Table 8.3, we get that,

$$\begin{aligned} E(Y^2) &= \left(2^2 \times \frac{1}{100}\right) + \left(3^2 \times \frac{4}{100}\right) + \ldots + \left(7^2 \times \frac{24}{100}\right) + \left(8^2 \times \frac{16}{100}\right) \\ &= \left(4 \times \frac{1}{100}\right) + \left(9 \times \frac{4}{100}\right) + \ldots + \left(49 \times \frac{24}{100}\right) + \left(64 \times \frac{16}{100}\right) \\ &= \frac{4}{100} + \frac{36}{100} + \frac{160}{100} + \frac{500}{100} + \frac{900}{100} + \frac{1176}{100} + \frac{1024}{100} \\ &= 38\,. \end{aligned} \tag{8.4.14}$$

Finally, we can substitute and calculate $\mathrm{Var}(Y)$,

$$\begin{aligned}\mathrm{Var}(Y) &= E(Y^2) - (E(Y))^2 \\ &= 38 - 6^2 \\ &= 2\,.\end{aligned} \tag{8.4.15}$$

(vi) Using Section 2.12, we can simplify this as,

$$E(3X + 2Y) = 3\,E(X) + 2\,E(Y) \tag{8.4.16}$$

Substituting the values we computed above,

$$\begin{aligned}E(3X + 2Y) &= 3 \times 3 + 2 \times 2 \\ &= 9 + 4 \\ &= 13\,.\end{aligned} \tag{8.4.17}$$

(b) We are given that a discrete random variable X follows a Poisson Distribution. From Def. 2.7.1, we can write,

$$X \sim \mathrm{Pois}(1.5)\,. \tag{8.4.18}$$

and

$$P(X = x) = \frac{1.5^x \times e^{-1.5}}{x!}\,. \tag{8.4.19}$$

Since X is discrete,

$$\begin{aligned}P(X \geq 2) &= 1 - P(X < 2) \\ &= 1 - (P(0) + P(1))\,.\end{aligned} \tag{8.4.20}$$

Using Eq. (8.4.19), we can calculate these probabilities,

$$\begin{aligned}P(0) &= \frac{1.5^0 \times e^{-1.5}}{0!} \\ &= \frac{1 \times e^{-1.5}}{1} \\ &= e^{-1.5}\,,\end{aligned} \tag{8.4.21}$$

and

$$\begin{aligned}P(1) &= \frac{1.5^1 \times e^{-1.5}}{1!} \\ &= \frac{1.5 \times e^{-1.5}}{1} \\ &= 1.5e^{-1.5}\,.\end{aligned} \tag{8.4.22}$$

Substituting these values into Eq. (8.4.20), we find,

$$\begin{aligned}P(X \geq 2) &= 1 - (P(0) + P(1)) \\ &= 1 - (e^{-1.5} + 1.5e^{-1.5}) \\ &= 1 - 2.5e^{-1.5} \\ &= 0.442\,.\end{aligned} \tag{8.4.23}$$

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Chapter 9

2009

9.1 Module 1: Managing Uncertainty

1. (a) We can construct the following activity network from the given information.

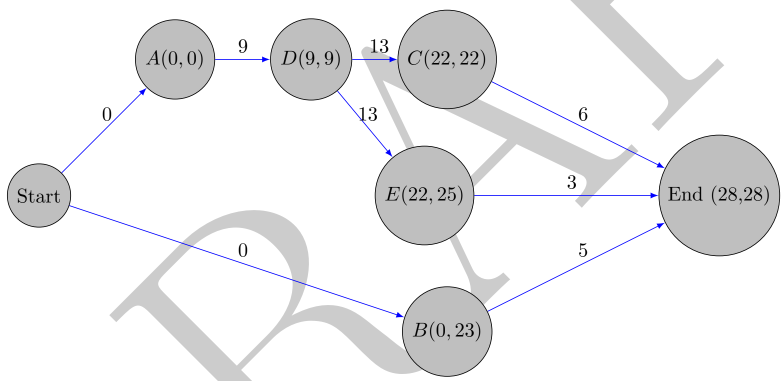


Figure 9.1: Showing the Activity Network of the project.

(b) We can analyse the activity network for the desired information as follows.

Activity	Earliest Start Time	Latest Start Time	Float Time
A	0	0	0
B	0	23	23
C	22	22	0
D	9	9	0
E	22	25	3

Table 9.1: Showing the start times and float times of the activities.

(c) **(i)** From Def. 1.2.7, the critical path is the path through the network with 0 float time,

$$\text{Start} \to A \to D \to C \to \text{End}\,. \tag{9.1.1}$$

(ii) The minimum completion time of the project is given by the duration of the critical path, 28 days.

2. **(a)** If $\boldsymbol{p}$ is the propositon "Tom works hard" and $\boldsymbol{q}$ is the proposition "Tom is successful", we can express $\boldsymbol{p} \iff \boldsymbol{q}$ in words as "Tom works hard if and only if Tom is successful."

(b) Given the statement $\boldsymbol{p} \implies \boldsymbol{q}$, we can write the:

(i) Inverse

$$\sim \boldsymbol{p} \implies \sim \boldsymbol{q}\;. \tag{9.2.1}$$

(ii) Converse

$$\boldsymbol{q} \implies \boldsymbol{p}\,. \tag{9.2.2}$$

(iii) Contrapositive

$$\sim \boldsymbol{q} \implies \sim \boldsymbol{p}\;. \tag{9.2.3}$$

(c) We can construct the following truth table for the proposition $(\boldsymbol{a} \wedge (\boldsymbol{a} \implies \boldsymbol{b})) \wedge \sim \boldsymbol{b}$.

$\boldsymbol{a}$	$\boldsymbol{b}$	$\sim \boldsymbol{b}$	$\boldsymbol{a} \implies \boldsymbol{b}$	$\boldsymbol{a} \wedge (\boldsymbol{a} \implies \boldsymbol{b})$	$(\boldsymbol{a} \wedge (\boldsymbol{a} \implies \boldsymbol{b})) \wedge \sim \boldsymbol{b}$
0	0	1	1	0	0
0	1	0	1	0	0
1	0	1	0	0	0
1	1	0	1	1	0

Table 9.2: Truth table of $(\boldsymbol{a} \wedge (\boldsymbol{a} \implies \boldsymbol{b})) \wedge \sim \boldsymbol{b}$.

From the truth table, it can be seen that the proposition $(\boldsymbol{a} \wedge (\boldsymbol{a} \implies \boldsymbol{b})) \wedge \sim \boldsymbol{b}$ is a contradiction because its truth value is always 0.

(d) From A to E, the possible paths are,

$$\begin{aligned}
&A \to E\,,\\
&A \to D \to E\,,\\
&A \to B \to D \to E\,,\\
&A \to B \to C \to E\,,\\
&A \to B \to C \to D \to E\,,\\
&A \to B \to D \to C \to E\,.
\end{aligned} \tag{9.2.4}$$

(e) The degree of vertex D is 4.

(f) The Boolean expression $(\boldsymbol{x} \wedge \boldsymbol{y}) \vee \sim \boldsymbol{y}$ can be expressed as the following logic gate circuit.

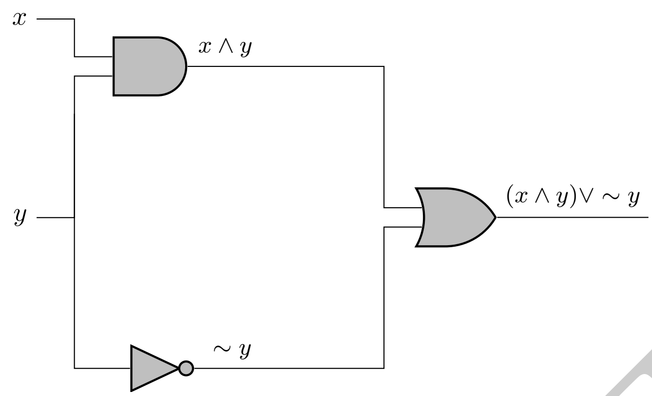


Figure 9.2: Logic gate equivalent of $\boldsymbol{(x \wedge y) \vee \sim y}$.

(g) **(i)** The Boolean expression for the given logic gate circuit is:

$$\boldsymbol{\sim (a \vee b)}\ . \tag{9.2.5}$$

(ii) The Boolean expression for the given logic gate circuit is:

$$\boldsymbol{\sim a \wedge \sim b}\ . \tag{9.2.6}$$

(iii) We can construct the following truth table to represent the circuits above.

$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{\sim a}$	$\boldsymbol{\sim b}$	$\boldsymbol{a \vee b}$	$\boldsymbol{\sim (a \vee b)}$	$\boldsymbol{\sim a \wedge \sim b}$
0	0	1	1	0	1	1
0	1	1	0	1	0	0
1	0	0	1	1	0	0
1	1	0	0	1	0	0

Table 9.3: Truth table of $\boldsymbol{\sim (a \vee b)}$ and $\boldsymbol{(\sim a \wedge \sim b)}$.

From Table 9.3, we see that $\boldsymbol{\sim (a \vee b)}$ and $\boldsymbol{(\sim a \wedge \sim b)}$ are logically equivalent.

9.2 Module 2: Probability and Distributions

3. **(a)** We are given 7 distinct letters.

(i) From Def. 2.1.6, we can find that the number of total arrangements using all 7 letters is,

$$\begin{aligned} {}^7P_7 &= \frac{7!}{(7-7)!} \\ &= \frac{7!}{(0)!} \\ &= 7! \\ &= 5040\,. \end{aligned} \tag{9.3.1}$$

(ii) There are 4 consonants and 3 vowels. In order to find the number or arrangements of interchanging consonants and vowels, we can form a line denoted with C_x and V_y, where C and V represent consonants and vowels respectively, and x and y represent the number of vowels and consonants remaining as follows,

$$C_4 : V_3 : C_3 : V_2 : C_2 : V_1 : C_1 \,. \tag{9.3.2}$$

Thus, from Def. 2.1.1, we can find the number of arrangements with alternating consonants and vowels as follows,

$$\begin{aligned} \text{Arrangements with alternating pattern} &= 4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1 \\ &= 144 \,. \end{aligned} \tag{9.3.3}$$

Thus, the probability that an alternating arrangemnet is chosen at random is,

$$P(\text{Alternating Sequence}) = \frac{144}{5040} = \frac{1}{35} \,. \tag{9.3.4}$$

(b) **(i)** Let L be the event that a taxi arrives late. Let X be the event of calling a taxi. We want to find

$$P(L) \,. \tag{9.3.5}$$

We are given that

$$\begin{aligned} P(X = F) &= 0.3 \\ P(X = G) &= 0.25 \\ P(X = T) &= 0.45 \end{aligned} \tag{9.3.6}$$

and

$$\begin{aligned} P(L|X = F) &= 0.03 \\ P(L|X = G) &= 0.05 \\ P(L|X = T) &= 0.10 \end{aligned} \tag{9.3.7}$$

As a taxi must be called, the probability that a taxi arrives late is,

$$P(L) = P((X = F) \wedge L) + P((X = G) \wedge L) + P((X = T) \wedge L) \,. \tag{9.3.8}$$

Thus, we can find $P(L)$ as follows,

$$\begin{aligned} P((X = F) \wedge L) &= 0.3 \times 0.03 \\ &= 0.009 \,. \end{aligned} \tag{9.3.9}$$

$$\begin{aligned} P((X = G) \wedge L) &= 0.25 \times 0.05 \\ &= 0.0125 \end{aligned} \tag{9.3.10}$$

$$P((X=T)\wedge L) = 0.45 \times 0.1$$
$$= 0.045\,. \tag{9.3.11}$$

$$\begin{aligned}P(L) &= P((X=F)\wedge L) + P((X=G)\wedge L) + P((X=T)\wedge L)\\ &= 0.009 + 0.0125 + 0.045\\ &= 0.0665 = 6.65\%\,.\end{aligned} \tag{9.3.12}$$

Alternatively, we know that,

$$P(L) = \sum_X P(L|X)P(X)\,. \tag{9.3.13}$$

Thus, we can compute $P(L)$ as,

$$\begin{aligned}P(L) &= P(L|X=F)P(X=F) + P(L|X=G)P(X=G) + P(L|X=T)P(X=T)\\ &= 0.3\times 0.03 + 0.25\times 0.05 + 0.45\times 0.1\\ &= 0.0665 = 6.65\%\,.\end{aligned} \tag{9.3.14}$$

(ii) We want to compute $P(X=T|L)$. From Def. 2.2.5, we have,

$$\begin{aligned}P(X=T|L) &= \frac{P(T\wedge L)}{P(L)}\\ &= \frac{0.045}{0.0665}\\ &= \frac{90}{133}\\ &\approx 67.7\%\end{aligned} \tag{9.3.15}$$

(c) The probability density function, f, of a continuous random variable is given,

(i) We want to compute $P(X<1)$. Note that,

$$P(1<X) = P(1<X<\infty)\,. \tag{9.3.16}$$

From Note 2.9.1,

$$P(1<X<\infty) = \int_1^\infty f(x)\,\mathrm{d}x \tag{9.3.17}$$

Next, we must split up the integral over the different intervals f is defined,

$$P(1<X) = \int_1^2 f(x)\,\mathrm{d}x + \int_2^\infty f(x)\,\mathrm{d}x\,. \tag{9.3.18}$$

Substituting,

$$
\begin{aligned}
P(1 < X) &= \int_1^2 \left(\frac{3}{8}(4 - 4x + x^2)\right) \mathrm{d}x + 0 \\
&= \frac{3}{8} \times \left[4x - 2x^2 + \frac{x^3}{3}\right]_1^2 \\
&= \frac{3}{8} \times \left(\left[4(2) - 2(2)^2 + \frac{(2)^3}{3}\right] - \left[4(1) - 2(1)^2 + \frac{(1)^3}{3}\right]\right) \\
&= \frac{3}{8} \times \left(\frac{8}{3} - \frac{7}{3}\right) \\
&= \frac{3}{8} \times \frac{1}{3} \\
&= \frac{1}{8}\,. \qquad (9.3.19)
\end{aligned}
$$

(ii) Using Def. 2.9.2, $E(X)$ is given by,

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, \mathrm{d}x\,. \qquad (9.3.20)$$

Again, we must split up the integral over the different regions,

$$E(X) = \int_{-\infty}^{0} x f(x)\mathrm{d}x + \int_0^2 x f(x)\mathrm{d}x + \int_2^{\infty} x f(x)\, \mathrm{d}x\,, \qquad (9.3.21)$$

before we can substitute and calculate,

$$
\begin{aligned}
E(X) &= 0 + \int_0^2 x f(x)\, \mathrm{d}x + 0 \\
&= \int_0^2 \left(x \times \frac{3}{8}(4 - 4x + x^2)\right) \mathrm{d}x \\
&= \frac{3}{8} \times \int_0^2 \left(4x - 4x^2 + x^3\right) \mathrm{d}x \\
&= \frac{3}{8} \times \left[2x^2 - \frac{4x^3}{3} + \frac{x^4}{4}\right]_0^2 \\
&= \frac{3}{8} \times \frac{4}{3} \\
&= \frac{1}{2}\,. \qquad (9.3.22)
\end{aligned}
$$

$$(9.3.23)$$

Similarly, using Def. 2.9.3, we can find $\mathrm{Var}(X)$ as,

$$\mathrm{Var}(X) = E(X^2) - E(X)^2\,. \qquad (9.3.24)$$

We proceed by finding $E(X^2)$ as,

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x\,. \tag{9.3.25}$$

Again, we must split up the integral over the different regions,

$$E(X^2) = \int_{-\infty}^{0} x^2 f(x)\,\mathrm{d}x + \int_{0}^{2} x^2 f(x)\,\mathrm{d}x + \int_{2}^{\infty} x^2 f(x)\,\mathrm{d}x\,, \tag{9.3.26}$$

before we can substitute and calculate,

$$\begin{aligned} E(X^2) &= 0 + \int_0^2 \left(x^2 \times \frac{3}{8}(4 - 4x + x^2)\right)\mathrm{d}x + 0 \\ &= \frac{3}{8} \times \int_0^2 \left(4x^2 - 4x^3 + x^4\right)\mathrm{d}x \\ &= \frac{3}{8} \times \left[\frac{4x^3}{3} - x^4 + \frac{x^5}{5}\right]_0^2 \\ &= \frac{3}{8} \times \frac{16}{15} \\ &= \frac{2}{5}\,. \end{aligned} \tag{9.3.27}$$

Hence, we can compute $\mathrm{Var}(X)$ by substituting as follows,

$$\begin{aligned} \mathrm{Var}(X) &= E(X^2) - E(X)^2 \\ &= \frac{2}{5} - \left(\frac{1}{2}\right)^2 \\ &= \frac{2}{5} - \frac{1}{4} \\ &= \frac{3}{20}\,. \end{aligned} \tag{9.3.28}$$

(iii) From Def. 2.9.4, we can compute the cumulative density function, $F(a)$, which is defined as $P(X < a)$, as follows,

$$F(a) = \int_{-\infty}^{a} f(x)\,\mathrm{d}x\,. \tag{9.3.29}$$

Since $F(q) = \frac{1}{4}$, we know $0 \le q \le 2$. Hence, we must split up the integral as

$$F(a) = \int_{-\infty}^{0} f(x)\,\mathrm{d}x + \int_{0}^{a} f(x)\,\mathrm{d}x\,, \tag{9.3.30}$$

before we can substitute and evaluate,

$$\begin{aligned}
F(a) &= 0 + \int_0^a f(x)\mathrm{d}x \\
&= \frac{3}{8} \times \int_0^a (2-x)^2 \mathrm{d}x \\
&= \frac{3}{8} \times \left[\left(\frac{(2-x)^3}{3} \times -1 \right) \right]_0^a \\
&= \frac{3}{8} \times \left(\frac{-(2-(a))^3}{3} - \frac{-(2-(0))^3}{3} \right) \\
&= \frac{3}{8} \times \left(\frac{-(2-a)^3}{3} - \frac{-8}{3} \right) \\
&= \frac{-(2-a)^3 + 8}{8} \\
&= 1 - \frac{(2-a)^3}{8}
\end{aligned} \tag{9.3.31}$$

Now, we can find q using the fact that

$$F(q) = \frac{1}{4}\,. \tag{9.3.32}$$

Hence

$$\begin{aligned}
1 - \frac{(2-q)^3}{8} &= \frac{1}{4} \\
8 - (2-q)^3 &= 2 \\
(2-q)^3 &= 6 \\
2 - q &= \sqrt[3]{6} \\
\implies q &= 2 - \sqrt[3]{6} \\
&\approx 0.183\,.
\end{aligned} \tag{9.3.33}$$

4. **(a)** We are given a scenario which illustrates a Binomial distribution with parameters, $p = 0.65$ and $n = 12$. Let X be the discrete random variable representing the number of people that pass the test,

$$X \sim \mathrm{Bin}(12, 0.65)\,. \tag{9.4.1}$$

(i) **a)** From, Def. 2.5.1, we know that,

$$P(X = x) = \binom{12}{x} (0.65)^x (1 - 0.65)^{12-x}\,. \tag{9.4.2}$$

Using this, we can find $P(X = 8)$,

$$\begin{aligned} P(X = 8) &= \binom{12}{8}(0.65)^8 \times (1 - 0.65)^{12-8} \\ &= \binom{12}{8}(0.65)^8 \times (0.35)^4 \\ &= 495 \times (0.65)^8 \times (0.35)^4 \\ &= 0.237 . \end{aligned} \tag{9.4.3}$$

b) Since X is discrete,

$$P(8 \leq X \leq 10) = P(8) + P(9) + P(10) . \tag{9.4.4}$$

Now we can use Eq. (9.4.2) to compute the additional $P(X = 9)$ and $P(X = 10)$,

$$\begin{aligned} P(9) &= \binom{12}{9}(0.65)^9 \times (1 - 0.65)^{12-9} \\ &= \binom{12}{9}(0.65)^9 \times (0.35)^3 \\ &= 220 \times (0.65)^9 \times (0.35)^3 \\ &= 0.195 , \end{aligned} \tag{9.4.5}$$

and,

$$\begin{aligned} P(10) &= \binom{12}{10}(0.65)^8 \times (1 - 0.65)^{12-10} \\ &= \binom{12}{10}(0.65)^8 \times (0.35)^2 \\ &= 66 \times (0.65)^{10} \times (0.35)^2 \\ &= 0.109 . \end{aligned} \tag{9.4.6}$$

Substituting these values into Eq. (9.4.4), we get that,

$$\begin{aligned} P(8 \leq X \leq 10) &= P(8) + P(9) + P(10) \\ &= 0.237 + 0.195 + 0.109 \\ &= 0.541 . \end{aligned} \tag{9.4.7}$$

(ii) For a Binomial distribution, the mean is given in Eq. (2.5.2) as,

$$E(X) = np . \tag{9.4.8}$$

Substituting, we find

$$\begin{aligned} E(X) &= 12 \times 0.65 \\ &= 7.8 . \end{aligned} \tag{9.4.9}$$

Similarly, using Eq. (2.5.3), we know the variance is given by,

$$\mathrm{Var} = npq = np(1-p)\,, \tag{9.4.10}$$

and so, we can compute the standard deviation as

$$\begin{aligned}\text{Standard Deviation} &= \sqrt{\text{Variance}}\\ &= \sqrt{12 \times 0.65 \times 0.35}\\ &= \sqrt{2.73}\\ &= 1.65\,.\end{aligned} \tag{9.4.11}$$

(iii) We will let Y be the discrete random variable which represents the number of days in which at least 8 but no more than 10 persons pass the test, over a 5 day period,

$$Y \sim \mathrm{Bin}(5, 0.541)\,. \tag{9.4.12}$$

From, Def. 2.5.1, we know that,

$$P(Y = y) = \binom{5}{y}(0.541)^y(1-0.541)^{5-y}\,. \tag{9.4.13}$$

Using Eq. (9.4.13), we can find that,

$$\begin{aligned}P(Y = 3) &= \binom{5}{3}(0.541)^3(1-0.541)^{5-3}\\ &= 10 \times (0.541)^3 \times (0.459)^2\\ &= 0.334\,.\end{aligned} \tag{9.4.14}$$

(b) Let X represent the number of accidents which occur in a week. X follows a Poisson distribution with parameter, $\lambda = 2$,

$$X \sim \mathrm{Pois}(2)\,. \tag{9.4.15}$$

From Def. 2.7.1, we know that,

$$P(X = x) = \frac{2^x \times e^{-2}}{x!}\,. \tag{9.4.16}$$

Since X is discrete, we know that,

$$P(X \geq 1) = 1 - P(X = 0)\,. \tag{9.4.17}$$

Using Eq. (9.4.16) we can find that,

$$\begin{aligned}P(X = 0) &= \frac{2^0 \times e^{-2}}{0!}\\ &= \frac{1 \times e^{-2}}{1}\\ &= e^{-2}\,.\end{aligned} \tag{9.4.18}$$

Substituting this into Eq. (9.4.16), we find,

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ &= 1 - e^{-2} \\ &= 0.865\,. \end{aligned} \tag{9.4.19}$$

(c) Let X represent the number of people that have a certain disease in a sample of 100 persons. X follows a Binomial distribution with parameters, $p = 0.04$ and $n = 100$,

$$X \sim \mathrm{Bin}(100, 0.04)\,. \tag{9.4.20}$$

Following Def. 2.8.1, we notice that,

$$\begin{aligned} n = 100 > 30\,, \\ \text{and } np = 100 \times 0.04 = 4 < 5\,. \end{aligned} \tag{9.4.21}$$

Thus, X can be approximated by a Poisson distribution with parameter, $\lambda = np = 4$,

$$X \sim \mathrm{Pois}(4)\,. \tag{9.4.22}$$

From Def. 2.7.1,

$$P(X = x) = \frac{4^x \times e^{-4}}{x!}\,. \tag{9.4.23}$$

Since X is discrete,

$$P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)\,. \tag{9.4.24}$$

We can determine the probabilities using Eq. (9.4.23),

$$\begin{aligned} P(X = 0) &= \frac{4^0 \times e^{-4}}{0!} \\ &= \frac{1 \times e^{-4}}{1} \\ &= e^{-4} \end{aligned} \tag{9.4.25}$$

$$\begin{aligned} P(X = 1) &= \frac{4^1 \times e^{-4}}{1!} \\ &= \frac{4 \times e^{-4}}{1} \\ &= 4e^{-4} \end{aligned} \tag{9.4.26}$$

$$\begin{aligned} P(X = 2) &= \frac{4^2 \times e^{-4}}{2!} \\ &= \frac{16 \times e^{-4}}{2} \\ &= 8e^{-4} \end{aligned} \tag{9.4.27}$$

$$
\begin{aligned}
P(X=3) &= \frac{4^3 \times e^{-4}}{3!} \\
&= \frac{64 \times e^{-4}}{6} \\
&= \frac{32e^{-4}}{3}\,. \qquad (9.4.28)
\end{aligned}
$$

Substituting these values into Eq. (9.4.24), we find that,

$$
\begin{aligned}
P(X \leq 3) &= P(X=0) + P(X=1) + P(X=2) + P(X=3) \\
&= e^{-4} \times \left(1 + 4 + 8 + \frac{32}{3}\right) \\
&= \frac{71e^{-4}}{3} \\
&= 0.433\,. \qquad (9.4.29)
\end{aligned}
$$

Chapter 10

2010

10.1 Module 1: Managing Uncertainty

1. (a) To maximize the objective function, $P = x + 2y$, we are given the following inequalities:

$$3x + 17y \leq 170\,, \tag{10.1.1}$$

$$7x + 8y \leq 175\,, \tag{10.1.2}$$

$$y \leq 9\,, \tag{10.1.3}$$

$$x \leq 2\,. \tag{10.1.4}$$

(i) The feasible region is shaded in the Fig. 10.1.

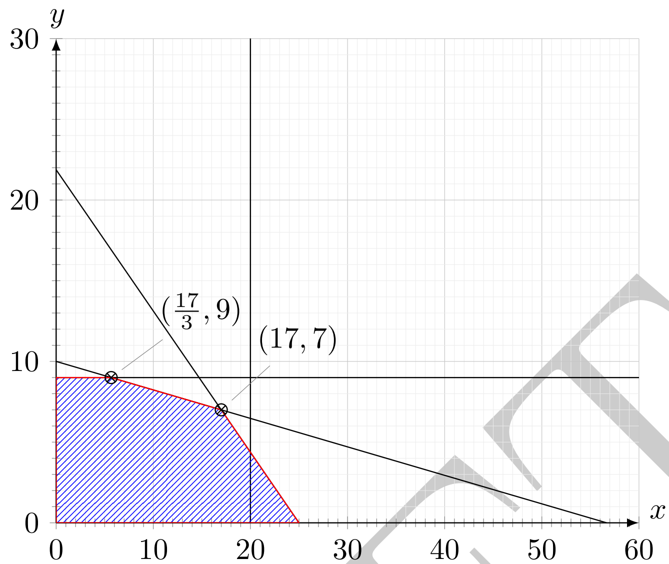


Figure 10.1: Linear Programming Graph.

(ii) We can solve the linear programming problem by performing a tour of the vertices (Def. **??**).

Vertice	$P = x + 2y$
(0, 0)	0
(0, 9)	18
($\frac{17}{3}$, 9)	23.67
(17, 7)	31
(25, 0)	25

Table 10.1: Tour of Vertices

Thus, from Table 10.1, the maximum value of P is 31, when $x = 17$ and $y = 7$.

(b) **(i)** We can determine which task should be assigned to each of five persons by applying the Hungarian algorithm as follows.

$$\begin{array}{ccccc} 50 & 52 & 51 & 54 & 53 \\ 55 & 53 & 50 & 50 & 52 \\ 49 & 51 & 48 & 53 & 50 \\ 50 & 50 & 52 & 50 & 51 \\ 52 & 54 & 49 & 52 & 54 \end{array}$$

Matrix From question

$$\begin{array}{ccccc} 0 & 2 & 1 & 4 & 3 \\ 5 & 3 & 0 & 0 & 2 \\ 1 & 3 & 0 & 5 & 2 \\ 0 & 0 & 2 & 0 & 1 \\ 3 & 5 & 0 & 3 & 5 \end{array}$$

Reducing Rows

$$\begin{array}{ccccc} 0 & 2 & 1 & 4 & 2 \\ 5 & 3 & 0 & 0 & 1 \\ 1 & 3 & 0 & 5 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 3 & 5 & 0 & 3 & 4 \end{array}$$

Reducing Columns

$$\begin{array}{ccccc} 0 & 2 & 1 & 4 & 2 \\ 5 & 3 & 0 & 0 & 1 \\ 1 & 3 & 0 & 5 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 3 & 5 & 0 & 3 & 4 \end{array}$$

Shading 0's

$$\begin{array}{ccccc} 0 & 2 & 3 & 4 & 2 \\ 5 & 3 & 2 & 0 & 1 \\ 0 & 2 & 0 & 4 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 2 & 4 & 0 & 2 & 3 \end{array}$$

Applying Step 5, 1.1.2

$$\begin{array}{ccccc} 0 & 2 & 3 & 4 & 2 \\ 5 & 3 & 2 & 0 & 1 \\ 0 & 2 & 0 & 4 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 2 & 4 & 0 & 2 & 3 \end{array}$$

Shading 0's

Table 10.2: Showing the steps of the Hungarian Algorithm.

From Table 10.2, the possible pairings for the tasks and persons are,

$$\begin{aligned} &\text{Person A} \to 1\,, \\ &\text{Person B} \to 4\,, \\ &\text{Person C} \to 1 \text{ or } 3 \text{ or } 5\,,\,, \\ &\text{Person D} \to 1 \text{ or } 2 \text{ or } 4 \text{ or } 5\,, \\ &\text{Person E} \to 3\,. \end{aligned} \tag{10.1.5}$$

Therefore, the optimum pairings are as follows:

$$\begin{aligned} &\text{Person A} \to 1\,, \\ &\text{Person B} \to 4\,, \\ &\text{Person C} \to 5\,, \\ &\text{Person D} \to 2\,, \\ &\text{Person E} \to 3\,. \end{aligned} \tag{10.1.6}$$

(ii) The minimum total time for the project is,

$$\text{Minimum Time} = 50 + 50 + 50 + 50 + 49 = 249 \text{ minutes}\,. \tag{10.1.7}$$

(c) Any of the following paths will start and finish at A, while visiting each node (apart from A) once. We recommend the simpler paths for a 3 mark question like this.

Visiting 1 vertex and returning to A:

$$A \to B \to A\,,$$
$$A \to E \to A\,,$$
$$A \to D \to A\,,$$

Visiting 2 vertices and returning to A:

$$A \to B \to E \to A\,,$$
$$A \to E \to B \to A\,,$$
$$A \to E \to D \to A\,,$$
$$A \to D \to E \to A\,,$$

Visiting 3 vertices and returning to A:

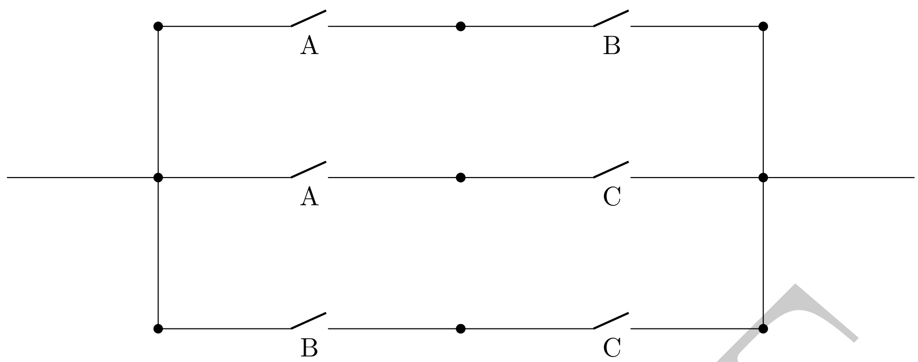
$$A \to B \to C \to E \to A\,,$$
$$A \to B \to C \to D \to A\,,$$
$$A \to E \to C \to B \to A\,,$$
$$A \to E \to C \to D \to A\,,$$
$$A \to D \to C \to E \to A\,,$$
$$A \to D \to C \to B \to A\,,$$

Visiting 4 vertices and returning to A:

$$A \to B \to C \to D \to E \to A\,,$$
$$A \to B \to C \to E \to D \to A\,,$$
$$A \to B \to E \to C \to D \to A\,,$$
$$A \to E \to B \to C \to D \to A\,,$$
$$A \to E \to D \to C \to B \to A\,,$$
$$A \to D \to C \to E \to B \to A\,,$$
$$A \to D \to C \to B \to E \to A\,,$$
$$A \to D \to E \to C \to B \to A\,,.$$

2. (a) Given that at least two members must vote yes for current to pass, we have to discard a simple $\boldsymbol{A} \vee \boldsymbol{B} \vee \boldsymbol{C}$ setup which will work even if **ONLY** one member votes yes. Similarly, a simple $\boldsymbol{A} \wedge \boldsymbol{B} \wedge \boldsymbol{C}$ setup must be discarded because it will only work if **ALL** three members vote yes. Hence, we can deduce that this circuit needs a combination of AND and OR setups.

Consider the three possible pairs of voters, $\boldsymbol{A} \wedge \boldsymbol{B}$, $\boldsymbol{A} \wedge \boldsymbol{C}$ and $\boldsymbol{B} \wedge \boldsymbol{C}$. If at least two members answer yes, then at least one of these pairs will be switched on. Coversely, if one of these pairs are switched on, then at least two voters must have voted yes. Therefore we can construct the switching circuit of $(\boldsymbol{A} \wedge \boldsymbol{B}) \vee (\boldsymbol{A} \wedge \boldsymbol{C}) \vee (\boldsymbol{B} \wedge \boldsymbol{C})$ below.



(b) We can construct the following truth table to show that the proposition

$$(\sim p \vee \sim q) \implies (p \wedge \sim q) \tag{10.2.1}$$

always takes the value of p.

p	q	$\sim p$	$\sim q$	$(\sim p \vee \sim q)$	$(p \wedge \sim q)$	$(\sim p \vee \sim q) \implies (p \wedge \sim q)$
0	0	1	1	1	0	0
0	1	1	0	1	0	0
1	0	0	1	1	1	1
1	1	0	0	0	0	1

Table 10.3: Showing the truth values of $(\sim p \vee \sim q) \implies (p \wedge \sim q)$

.

(c) We can draw a logic gate circuit for $a \implies b$, using the given equivalence statement that $a \implies b \equiv \sim a \vee b$.

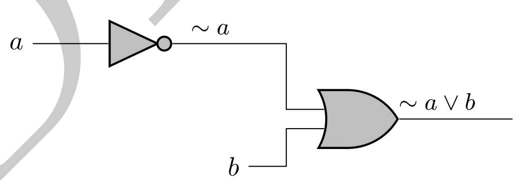


Figure 10.2: Showing the logic gate equivalent of $\sim a \vee b$.

(d) **(i)** We can complete the table given for the activity network as shown below.

Activity	Earliest Start Time	Latest Start Time	Float Time
A	0	0	0
B	6	6	0
C	16	16	0
D	6	17	11
E	18	19	1
F	18	18	0
G	22	22	0

Table 10.4: Float time of the activities.

(ii) From Table 10.4 and using Def. 1.2.6 and Def. 1.2.7, we can determine the critical path by looking at the path which has 0 float time:

$$\text{Start} \to A \to B \to C \to F \to G \to \text{Finish}\,. \tag{10.2.2}$$

10.2 Module 2: Probability and Distributions

3. (a) The cumulative density function, F, of some continuous random variable X is given.

(i) From Def. 2.9.1, recall that,

$$F(x) = P(X \leq x)\,. \tag{10.3.1}$$

From the given $F(x)$, we know that,

$$F(6) = 1\,. \tag{10.3.2}$$

Substituting in the given expression, we find,

$$\begin{aligned} F(6) = k(x-3) &= 1 \\ k(6-3) &= 1 \\ 3k &= 1 \\ \implies k &= \frac{1}{3}\,. \end{aligned} \tag{10.3.3}$$

Fig. 10.3 shows $y = F(x)$.

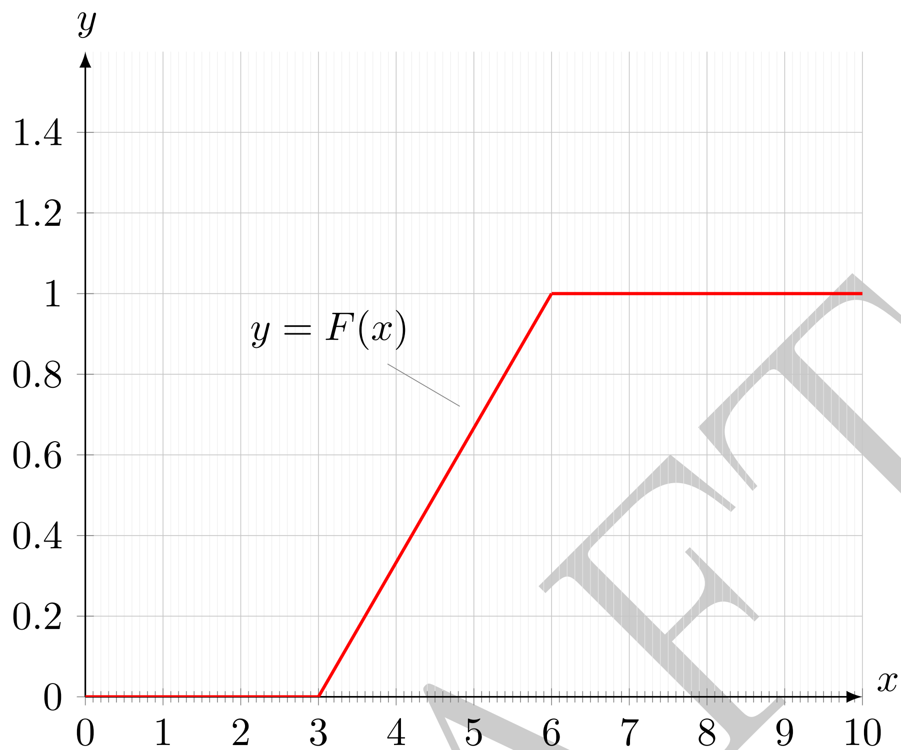


Figure 10.3: Graph of $y = F(x)$.

(ii) We can compute probabilites from the cumulative density function using Note 2.9.3,

$$P(3.5 \le X \le 5) = F(5) - F(3.5)\,. \tag{10.3.4}$$

Substituting, we find,

$$\begin{aligned} P(3.5 \le X \le 5) &= \frac{1}{3}(5-3) - \frac{1}{3}(3.5-3) \\ &= \frac{2}{3} - \frac{1}{6} \\ &= \frac{1}{2}\,. \end{aligned} \tag{10.3.5}$$

(iii) Recall Prop. 2.9.3,that the median M, is the value $x = M$ such that,

$$F(M) = \frac{1}{2}\,. \tag{10.3.6}$$

Thus, the median of X can be found as,

$$
\begin{aligned}
F(M) = \frac{1}{3}(M-3) &= \frac{1}{2} \\
\implies (M-3) &= 3 \times \frac{1}{2} \\
\implies M &= \frac{3}{2} + 3 \\
&= \frac{9}{2}\,.
\end{aligned} \tag{10.3.7}
$$

(iv) From Prop. 2.9.3, we know that, the LQ is the value of $x = LQ$ such that,

$$F(LQ) = \frac{1}{4}\,. \tag{10.3.8}$$

Thus, the lower quartile of X can be found as,

$$
\begin{aligned}
F(LQ) = \frac{1}{3}(LQ-3) &= \frac{1}{4} \\
\implies (LQ-3) &= 3 \times \frac{1}{4} \\
\implies LQ &= \frac{3}{4} + 3 \\
&= \frac{15}{4}\,.
\end{aligned} \tag{10.3.9}
$$

(b) (i) From Def. 2.9.4, we can relate the probability density function, f, from the cumulative distribution function, F, via,

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x} F(x)\,. \tag{10.3.10}$$

We need to calculate this in the different intervals over which F is defined. The only non-trivial one occurs for $3 \leq x \leq 6$,

$$
\begin{aligned}
f(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{3}(x-3)\right) \\
&= \frac{1}{3}\,.
\end{aligned} \tag{10.3.11}
$$

Otherwise,

$$f = \frac{\mathrm{d}}{\mathrm{d}x} F = 0\,. \tag{10.3.12}$$

Fig. 10.4 shows $y = f(x)$.

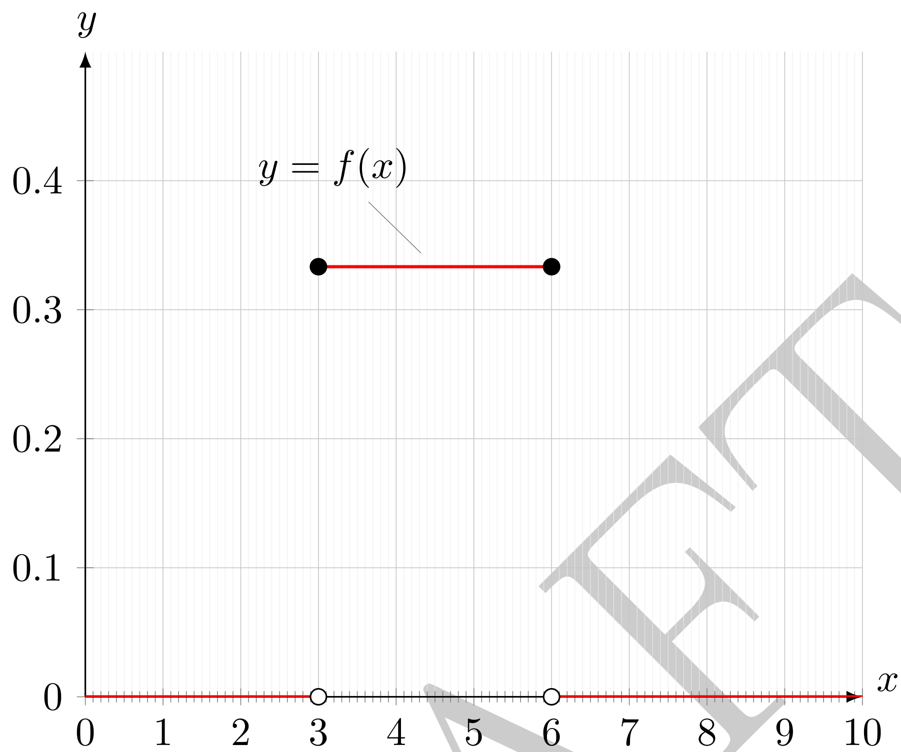


Figure 10.4: Graph of $y = f(x)$.

Summarizing,

$$f(x) = \begin{cases} 0, & x < 3 \\ \frac{1}{3}, & 3 \leq x \leq 6 \\ 0, & x > 6 \end{cases} \tag{10.3.13}$$

(ii) Using Def. 2.9.2, $E(X)$ is given as,

$$E(X) = \int_{-\infty}^{\infty} x f(x) \mathrm{d}x \,. \tag{10.3.14}$$

First, we must split up the intergral into the appropriate regions,

$$E(X) = \int_{-\infty}^{3} x f(x)\, \mathrm{d}x + \int_{3}^{6} x f(x)\, \mathrm{d}x + \int_{6}^{\infty} x f(x)\, \mathrm{d}x \,, \tag{10.3.15}$$

before we can evaluate,

$$\begin{aligned} E(X) &= 0 + \int_3^6 \left(x \times \frac{1}{3}\right) \mathrm{d}x + 0 \\ &= \left[\frac{x^2}{6}\right]_3^6 \\ &= \left(\frac{6^2}{6}\right) - \left(\frac{3^2}{6}\right) \\ &= 6 - \frac{9}{6} \\ &= 4.5\,. \end{aligned} \tag{10.3.16}$$

(iii) Using Def. 2.9.3, we know that Var(X) is given by,

$$\mathrm{Var}(X) = E(X^2) - E(X)^2\,. \tag{10.3.17}$$

We then proceed by first finding $E(X^2)$,

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x\,. \tag{10.3.18}$$

Again, the integral must be split up as,

$$E(X^2) = \int_{-\infty}^{3} f(x)\,\mathrm{d}x + \int_3^6 f(x)\,\mathrm{d}x + \int_6^{\infty} f(x)\,\mathrm{d}x\,, \tag{10.3.19}$$

before we can substitute and evaluate as,

$$\begin{aligned} E(X^2) &= 0 + \int_3^6 \left(x^2 \times \frac{1}{3}\right) \mathrm{d}x + 0 \\ &= \left[\frac{x^3}{9}\right]_3^6 \\ &= \left(\frac{6^3}{9}\right) - \left(\frac{3^3}{9}\right) \\ &= 24 - 3 = 21\,. \end{aligned} \tag{10.3.20}$$

Finally, substituting our values into Eq. (10.3.17), we can determine $\mathrm{Var}(X)$ as,

$$\begin{aligned} \mathrm{Var}(X) &= E(X^2) - E(X)^2 \\ &= 21 - 4.5^2 \\ &= \frac{3}{4}\,. \end{aligned} \tag{10.3.21}$$

4. (a) Let X be the continuous random variable representing the lengths of wood produced,

$$X \sim N(3, \sigma^2)\,. \tag{10.4.1}$$

We are also given that,

$$P(X < 2.75) = 0.15\,. \tag{10.4.2}$$

(i) We can use the information given to determine σ. We begin with standardizing the random variable X to the normal random variable Z as follows,

$$\begin{aligned} P(X < 2.75) = P\left(\frac{X-\mu}{\sigma} < \frac{2.75-\mu}{\sigma}\right) &= 0.15 \\ P\left(Z < \frac{2.75-3}{\sigma}\right) &= 0.15 \\ P\left(Z < \frac{-0.25}{\sigma}\right) &= 0.15 \\ \implies \Phi\left(\frac{-0.25}{\sigma}\right) &= 0.15\,. \end{aligned} \tag{10.4.3}$$

From our Z-tables, we see that $\Phi(-1.04) = 0.15$. Thus,

$$\begin{aligned} \frac{-0.25}{\sigma} &= -1.04 \\ \implies \sigma &= \frac{-0.25}{-1.04} \\ &\approx 0.24\,. \end{aligned} \tag{10.4.4}$$

(ii) We can update our distritbuion with the new information,

$$X \sim N(3, 0.24^2)\,. \tag{10.4.5}$$

To determine $P(X > 3.2)$, we can proceed by standardizing standardizing X to Z as follows,

$$\begin{aligned} P(X > 3.2) &= P\left(\frac{X-\mu}{\sigma} > \frac{3.2-\mu}{\sigma}\right) \\ &= P\left(Z > \frac{3.2-3}{0.24}\right) \\ &= P\left(Z > \frac{0.2}{0.24}\right) \\ &= 1 - \Phi\left(\frac{5}{6}\right) \\ &= 1 - 0.798 \\ &= 0.202\,. \end{aligned} \tag{10.4.6}$$

(b) **(i)** The hypotheses for this test are,

- H_0: These is no association between the number of employees working overtime and the distance from their home.
- H_1: These is an association between the number of employees working overtime and the distance from their home.

(ii) We should note that, if there is no association between the number of employees working overtime and the distance from their home, the expected value for the number of employees working overtime would be 20 ($\frac{\text{Total}=100}{5}$).

(iii) We are asked to perform a χ^2 goodness-of-fit test at the 5% significance level. From Section 2.13, we know that,

$$\chi^2_{\text{calc.}} = \sum_{\forall k} \left(\frac{(O_k - E_k)^2}{E_k} \right) . \tag{10.4.7}$$

We summarize the details of this calculation in Table 10.5.

Distances	Observed Employees, O	Expected Employees, E	$\frac{(O-E)^2}{E}$
10-14	30	20	5
15-19	25	20	1.25
20-24	14	20	1.8
25-29	19	20	0.05
30-34	12	20	3.2
Total, $\chi^2_{\text{calc.}}$			11.3

Table 10.5: χ^2 Table.

The number of degrees of freedom, ν, is,

$$\nu = \text{Number of classes} - 1 = 5 - 1 = 4 . \tag{10.4.8}$$

At the 5% significance level, we that we can reject H_0 if and only if,

$$\begin{aligned} &\chi^2_{\text{calc.}} > \chi^2_{\alpha}(\nu) \\ \Longrightarrow\ &\chi^2_{\text{calc.}} > \chi^2_{0.05}(4) \\ \Longrightarrow\ &\chi^2_{\text{calc.}} > 9.488 . \end{aligned} \tag{10.4.9}$$

Therefore, from Table 10.5, since $11.3 > 9.488$, we reject H_0.

Chapter 11

2011

11.1 Module 1: Managing Uncertainty

1. **(a)** We can construct the following truth table to evaluate $(\sim \boldsymbol{p} \vee \sim \boldsymbol{q}) \implies (\boldsymbol{p} \wedge \sim \boldsymbol{q})$

$\boldsymbol{p}$	$\boldsymbol{q}$	$\sim \boldsymbol{p}$	$\sim \boldsymbol{q}$	$(\sim \boldsymbol{p} \vee \sim \boldsymbol{q})$	$(\boldsymbol{p} \wedge \sim \boldsymbol{q})$	$(\sim \boldsymbol{p} \vee \sim \boldsymbol{q}) \implies (\boldsymbol{p} \wedge \sim \boldsymbol{q})$
0	0	1	1	1	0	0
0	1	1	0	1	0	0
1	0	0	1	1	1	1
1	1	0	0	0	0	1

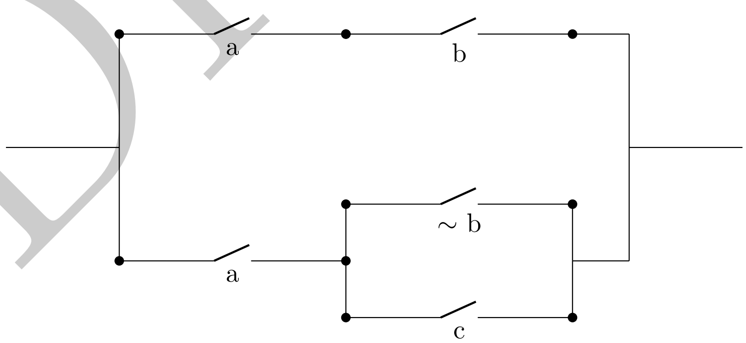
Table 11.1: Truth Table of $(\sim \boldsymbol{p} \vee \sim \boldsymbol{q}) \implies (\boldsymbol{p} \wedge \sim \boldsymbol{q})$.

We can see from Table 11.1 the proposition always takes the value of $\boldsymbol{p}$.

(b) We can write the following Boolean expression for the given logic circuit.

$$(\boldsymbol{p} \wedge (\boldsymbol{q} \vee \sim \boldsymbol{r})) \vee \boldsymbol{r}\,. \tag{11.1.1}$$

(c) We can draw the following switching circuit to represent $(\boldsymbol{a} \wedge \boldsymbol{b}) \vee (\boldsymbol{a} \wedge (\sim \boldsymbol{b} \vee \boldsymbol{c}))$ below.



(d) **(i)** Given that $\boldsymbol{p}$ is the proposition "There is a west wind" and $\boldsymbol{q}$ is the proposition "We shall have rain",

a) "If there is a west wind, then we shall have rain" can be expressed as,

$$\boldsymbol{p} \implies \boldsymbol{q}\,. \tag{11.1.2}$$

b) "If there is no rain, then the west wind does not blow" can be expressed as,

$$\sim \boldsymbol{q} \implies \sim \boldsymbol{p}\ . \tag{11.1.3}$$

(ii) We can construct the following truth table for the two previous logic statements.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\sim \boldsymbol{p}$	$\sim \boldsymbol{q}$	$\boldsymbol{p} \implies \boldsymbol{q}$	$\sim \boldsymbol{q} \implies \sim \boldsymbol{p}$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	0	0	1	1

Table 11.2: Truth values of $\boldsymbol{p} \implies \boldsymbol{q}$ and $\sim \boldsymbol{q} \implies \sim \boldsymbol{p}$.

(e) Using De Morgan's Law (Def. 1.3.6), we can see that,

$$\sim ((\boldsymbol{p} \wedge \boldsymbol{q}) \vee \sim \boldsymbol{p}) \equiv \ \sim (\boldsymbol{p} \wedge \boldsymbol{q}) \wedge \boldsymbol{p}\ . \tag{11.1.4}$$

From De Morgan's Law (Def. 1.3.6), we also know that,

$$\sim (\boldsymbol{p} \wedge \boldsymbol{q}) \equiv (\sim \boldsymbol{p} \vee \sim \boldsymbol{q})\ . \tag{11.1.5}$$

Therefore, Eq. (11.1.4) becomes,

$$\sim ((\boldsymbol{p} \wedge \boldsymbol{q}) \vee \sim \boldsymbol{p}) \equiv (\sim \boldsymbol{p} \vee \sim \boldsymbol{q}) \wedge \boldsymbol{p}\ . \tag{11.1.6}$$

Using Distributive (Def. 1.3.9), Complement (Def. 1.3.4) and Identity (Def. 1.3.2) laws,

$$\begin{aligned}(\sim \boldsymbol{p} \vee \sim \boldsymbol{q}) \wedge \boldsymbol{p} &\equiv (\boldsymbol{p} \wedge \sim \boldsymbol{p}) \vee (\boldsymbol{p} \wedge \sim \boldsymbol{q}) \\ &\equiv (\boldsymbol{F}) \vee (\boldsymbol{p} \wedge \sim \boldsymbol{q}) \\ &\equiv (\boldsymbol{p} \wedge \sim \boldsymbol{q}) \\ &\equiv (\sim \boldsymbol{q} \wedge \boldsymbol{p})\ . \end{aligned} \tag{11.1.7}$$

Thus,

$$\sim ((\boldsymbol{p} \wedge \boldsymbol{q}) \vee \sim \boldsymbol{p}) \equiv (\sim \boldsymbol{q} \wedge \boldsymbol{p})\ . \tag{11.1.8}$$

2. (a) We can allocate runners to a position to minimise the sum of their average times by applying the Hungarian algorithm on the given data.
From Table 11.3, we can see that the possible pairings for the runners are as follows:

$$\begin{aligned}&\text{Person A} \to 1\,,\\ &\text{Person B} \to 3,\ 4\,,\\ &\text{Person C} \to 3\,,\\ &\text{Person D} \to 1,\ 2,\ 4\,.\end{aligned} \tag{11.2.1}$$

$$\begin{array}{cccc} 13 & 15 & 14 & 17 \\ 18 & 16 & 13 & 13 \\ 12 & 14 & 11 & 16 \\ 13 & 13 & 15 & 13 \end{array}$$

Matrix From question

$$\begin{array}{cccc} 0 & 2 & 1 & 4 \\ 5 & 3 & 0 & 0 \\ 1 & 3 & 0 & 5 \\ 0 & 0 & 2 & 0 \end{array}$$

Reducing Rows

$$\begin{array}{cccc} 0 & 2 & 1 & 4 \\ 5 & 3 & 0 & 0 \\ 1 & 3 & 0 & 5 \\ 0 & 0 & 2 & 0 \end{array}$$

Reducing Columns

$$\begin{array}{cccc} 0 & 2 & 1 & 4 \\ 5 & 3 & 0 & 0 \\ 1 & 3 & 0 & 5 \\ 0 & 0 & 2 & 0 \end{array}$$

Shading 0's

Table 11.3: Showing the steps of the Hungarian Algorithm.

Therefore, the matchings to minimize the total time are,

$$\begin{aligned} &\text{Person A} \to 1\,, \\ &\text{Person B} \to 4\,, \\ &\text{Person C} \to 3\,, \\ &\text{Person D} \to 2\,. \end{aligned} \tag{11.2.2}$$

Therefore, the minimum time for the entire race is,

$$\text{Total time} = 50\,\text{seconds}\,. \tag{11.2.3}$$

(b) **(i)** We can construct the following activity network for the project from the given data.

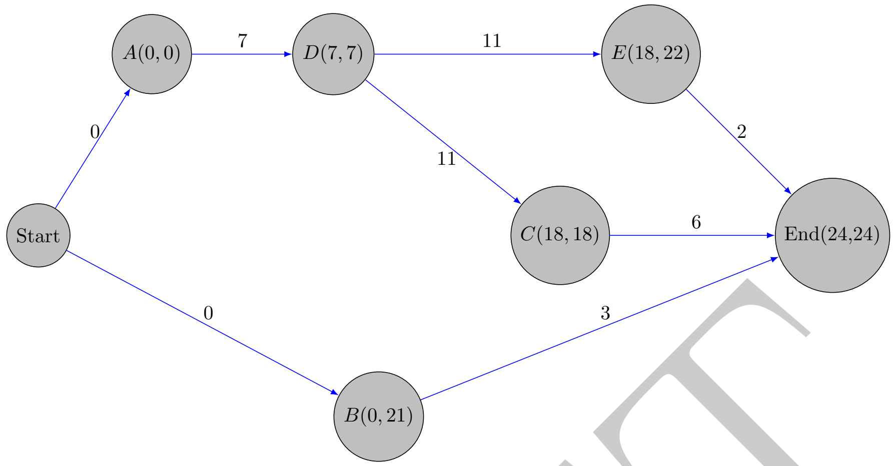


Figure 11.1: Activity Network of the operation.

(ii) We can complete the given table as follows.

Activity	Earliest Start Time	Latest Start Time	Float Time
A	0	0	0
B	0	21	21
C	18	18	0
D	7	7	0
E	18	22	4

Table 11.4: Float times of the activities.

(iii) **a)** From Def. 1.2.7, the critical path can be determined as the path with 0 float time:

$$\text{Start} \to A \to D \to C \to \text{End}\,. \tag{11.2.4}$$

b) The minimum completion time of this project, given by the length of the critical path, is 24 hours.

11.2 Module 2: Probability and Distributions

3. **(a)** We make some observations about the word MAXIMUM: There are 5 letters which make up the word, {A, I , M, U, X}, and all of the letters occur exactly once, except for M which occurs 3 times. The number of letters in the word, n, is equal to 7.

(i) From Def. 2.1.7, the number of permutations of seven elements when three are identical is,

$$\begin{aligned}
\text{Number of Arrangements} &= \frac{7!}{3!} \\
&= 7 \times 6 \times 5 \times 4 \\
&= 840\,. \qquad (11.3.1)
\end{aligned}$$

(ii) We can break this down into a two step process. Note that, to arrange the letters such that the last letter is a vowel, we must first pick the last letter to be a vowel, and then arrange the first six letters. So, according to Def. 2.1.1,

$$\begin{aligned}
\#\text{arrangements with last letter a vowel} = \#&\text{ways to pick one vowel} \\
&\times \\
\#&\text{of ways to arrange the first six letters}
\end{aligned} \qquad (11.3.2)$$

Following this strategy, we note that there are three distinct vowels { A, I, U}. Thus,

$$\#\text{ of ways to pick one vowel } = 3\,. \qquad (11.3.3)$$

After we pick one vowel to be the last letter, we can add the other two vowels back into the set of the first six letters that preceed the last letter. The first six letters have three M's. So, using Def. 2.1.7 for arrangements with identical elements,

$$\#\text{ of ways to arrange first six letters } = \frac{6!}{3!}\,. \qquad (11.3.4)$$

Substituting these in the above, we find that

$$\begin{aligned}
\#\text{arrangements with last letter a vowel} &= 3 \times \frac{6!}{3!} \\
&= 120 \times 3 \\
&= 360\,. \qquad (11.3.5)
\end{aligned}$$

(iii) The strategy now is to treat the three M's as a single item. Now, there are only 5 items to sort {A, I, U, X, and 1 set of 3 M's}. Thus, from Def. 2.1.6,

$$\begin{aligned}
{}^5P_5 &= \frac{5!}{(5-5)!} \\
&= \frac{5!}{0!} \\
&= 5! \\
&= 120\,. \qquad (11.3.6)
\end{aligned}$$

(b) We should notice that a committee can only have more that 3 women in it if ALL 4 women are chosen. Therefore, there is only 1 possible committee with all 4 women.

Thus, we can find the total number of committees that can be formed among all the persons, and minus the 1 committee with 4 women, in order to find the number of committees with AT MOST 3 women. Thus,

$$\text{Number of committees with AT MOST 3 Women} = \text{Total number of committees} - 1\,. \tag{11.3.7}$$

Since every possible person out of the 10 is distinct, we can use Def. 2.1.4 as follows,

$$\begin{aligned}\text{Number of Committees} =&\ {}^{10}C_4 \\ =&\ \frac{10!}{4! \times (10-4)!} \\ =&\ \frac{10!}{4! \times 6!} \\ =&\ 210\,.\end{aligned} \tag{11.3.8}$$

Thus,

$$\text{Number of committees with at most 3 Women} = 210 - 1 = 209\,. \tag{11.3.9}$$

(c) Let X be the discrete random variable which representing the number attempts needed to draw a green ball.

(i) Because a ball is returned to the box if it is not red, the probability of success and failure remain constant with time. From Section 2.6, we see that

$$X \sim \text{Geo}\left(\frac{3}{9}\right) \text{ or } X \sim \text{Geo}\left(\frac{1}{3}\right)\,. \tag{11.3.10}$$

(ii) **a)** From Def. 2.6.1,

$$\begin{aligned}P(X = x) &= \frac{1}{3} \times \left(1 - \frac{1}{3}\right)^{x-1} \\ &= \frac{1}{3} \times \left(\frac{2}{3}\right)^{x-1}\,.\end{aligned} \tag{11.3.11}$$

Thus, we can find $P(X = 2)$ as follows,

$$\begin{aligned}P(X = 2) &= \frac{1}{3} \times \left(\frac{2}{3}\right)^{2-1} \\ &= \frac{1}{3} \times \frac{2}{3} \\ &= \frac{2}{9}\,.\end{aligned} \tag{11.3.12}$$

b) We want to find $P(X \geq 3)$. Since X is discrete,

$$P(X \geq 3) = P(X > 2)\,. \tag{11.3.13}$$

From Def. 2.6.2,

$$\begin{aligned} P(X > 2) &= (1-p)^2 \\ &= \left(\frac{2}{3}\right)^2 \\ &= \frac{4}{9}\,. \end{aligned} \tag{11.3.14}$$

Alternatively, we could note

$$\begin{aligned} P(X \geq 3) &= 1 - P(X < 2) \\ &= 1 - (P(X = 1) + P(X = 2)) \end{aligned} \tag{11.3.15}$$

and compute

$$\begin{aligned} P(X = 1) &= \frac{1}{3} \times \left(\frac{2}{3}\right)^{1-1} \\ &= \frac{1}{3}\,. \end{aligned} \tag{11.3.16}$$

Since we have $P(X = 2)$ from above, we can substitute to find $P(X \geq 3)$,

$$\begin{aligned} P(X \geq 3) &= 1 - \left(\frac{1}{3} + \frac{2}{9}\right) \\ &= \frac{4}{9}\,. \end{aligned} \tag{11.3.17}$$

c) We want to find $P(X < 4)$. Since X is discrete, we know that,

$$\begin{aligned} P(X < 4) &= 1 - P(X \geq 4) \\ &= 1 - P(X > 3) \end{aligned} \tag{11.3.18}$$

Using Def. 2.6.2, we can calculate $P(X > 3)$ as,

$$\begin{aligned} P(X > 3) &= (1-p)^3 \\ &= \left(\frac{2}{3}\right)^3 \\ &= \frac{8}{27}\,. \end{aligned} \tag{11.3.19}$$

Thus, $P(X < 4)$ can be calculated as follows,

$$\begin{aligned} P(X < 4) &= 1 - P(X > 3) \\ &= 1 - \frac{8}{27} \\ &= \frac{19}{27}\,. \end{aligned} \tag{11.3.20}$$

(iii) From Eq. (2.6.2), we can compute $E(X)$ of a geometric distribution as,

$$E(X) = \frac{1}{p}\,. \tag{11.3.21}$$

Substituting, we find

$$\begin{aligned} E(X) &= \frac{1}{\frac{1}{3}} \\ &= 3\,. \end{aligned} \tag{11.3.22}$$

4. (a) Let X be the discrete random variable representing the amount of accidents which occur in one week. X follows a Poisson distribution,

$$X \sim \text{Pois}(3)\,. \tag{11.4.1}$$

(i) From Def. 2.7.1, we know that,

$$P(X = x) = \frac{3^x \times e^{-3}}{x!}\,. \tag{11.4.2}$$

Using this, we can determine $P(X = 0)$,

$$\begin{aligned} P(X = 0) &= \frac{3^0 \times e^{-3}}{0!} \\ &= \frac{1 \times e^{-3}}{1} \\ &= \frac{1}{e^3}\,. \end{aligned} \tag{11.4.3}$$

(ii) To determine $P(X > 3)$ it is more convenient to write it as,

$$P(X > 3) = 1 - P(X \leq 3)\,. \tag{11.4.4}$$

Further, since X is discrete,

$$P(X > 3) = 1 - [P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)]\,. \tag{11.4.5}$$

We use Eq. (11.4.2) to determine the necessary probabilities,

$$P(X = 0) = \frac{1}{e^3}\,, \tag{11.4.6}$$

$$\begin{aligned} P(X = 1) &= \frac{3^1 \times e^{-3}}{1!} \\ &= \frac{3 \times e^{-3}}{1} \\ &= \frac{3}{e^3}\,, \end{aligned} \tag{11.4.7}$$

$$\begin{aligned} P(X=2) &= \frac{3^2 \times e^{-3}}{2!} \\ &= \frac{9 \times e^{-3}}{2} \\ &= \frac{9}{2e^3}\,, \end{aligned} \tag{11.4.8}$$

and

$$\begin{aligned} P(X=3) &= \frac{3^3 \times e^{-3}}{3!} \\ &= \frac{27 \times e^{-3}}{6} \\ &= \frac{27}{6e^3}\,. \end{aligned} \tag{11.4.9}$$

Substituting these into Eq. (11.4.5), we get,

$$\begin{aligned} P(X>3) &= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)] \\ &= 1 - \left(\frac{1}{e^3} + \frac{3}{e^3} + \frac{9}{2e^3} + \frac{27}{6e^3}\right) \\ &= 1 - \frac{13}{e^3} \\ &= 0.353\,. \end{aligned} \tag{11.4.10}$$

(iii) If there are on average 3 accidents in 1 week, there will be on average 6 accidents during a fortnight (2 week period). Let Y be the discrete random variable representing the number of accidents in a 2 week period,

$$Y \sim \text{Pois}(6)\,. \tag{11.4.11}$$

From Def. 2.7.1, we know that,

$$P(Y=y) = \frac{6^y \times e^{-6}}{y!}\,. \tag{11.4.12}$$

Thus, we can find $P(Y=6)$ as,

$$\begin{aligned} P(Y=6) &= \frac{6^6 \times e^{-6}}{6!} \\ &= \frac{46656 \times e^{-6}}{720} \\ &= \frac{324}{5 \times e^6} \\ &= 0.161\,. \end{aligned} \tag{11.4.13}$$

(b) We are given that the probability of success, $p = 0.84$. Let X be the discrete random variable representing the number of students that pass an examination. X follows a Binomial distribution,

$$X \sim \text{Bin}(0.84)\,. \tag{11.4.14}$$

From Def. 2.5.1, we know that we can determine $P(X = x)$ as,

$$P(X = x) = \binom{n}{x}(0.84)^x(1 - 0.84)^{n-x}\,. \tag{11.4.15}$$

Since two-thirds of 12 is 8, we want to find,

$$P(X > 8)\,. \tag{11.4.16}$$

Since X is discrete, we know that,

$$P(X > 8) = P(X = 9) + P(X = 10) + P(X = 11) + P(X = 12)\,. \tag{11.4.17}$$

Using Eq. (11.4.15), we can find each of these probabilities,

$$\begin{aligned} P(X = 9) &= \binom{12}{9} \times (0.84)^9 \times (1 - 0.84)^{12-9} \\ &= 220 \times (0.84^9) \times (0.16)^3 \\ &= 0.188\,, \end{aligned} \tag{11.4.18}$$

$$\begin{aligned} P(X = 10) &= \binom{12}{10} \times (0.84)^{10} \times (1 - 0.84)^{12-10} \\ &= 66 \times (0.84)^{10} \times (0.16)^2 \\ &= 0.296\,, \end{aligned} \tag{11.4.19}$$

$$\begin{aligned} P(X = 11) &= \binom{12}{11} \times (0.84)^{11} \times (1 - 0.84)^{12-11} \\ &= 12 \times (0.84)^{11} \times 0.16 \\ &= 0.282\,, \end{aligned} \tag{11.4.20}$$

and,

$$\begin{aligned} P(X = 12) &= \binom{12}{12} \times (0.84)^{12} \times (1 - 0.84)^{12-12} \\ &= 1 \times (0.84)^{12} \times 1 \\ &= 0.123\,. \end{aligned} \tag{11.4.21}$$

Hence, substituting these into Eq. (11.4.17), we get that,

$$\begin{aligned} P(X > 8) &= P(X = 9) + P(X = 10) + P(X = 11) + P(X = 12) \\ &= 0.188 + 0.296 + 0.282 + 0.123 \\ &= 0.889\,. \end{aligned} \tag{11.4.22}$$

(c) **(i)** The probability that the first marble drawn is white is $\frac{4}{9}$. The probability that the second marble drawn is black is $\frac{3}{8}$. The probability that the third marble drawn is red is $\frac{2}{7}$. Thus, from Def. 2.1.1,

$$\begin{aligned} P(\textit{white} \to \textit{black} \to \textit{red}) &= \frac{4}{9} \times \frac{3}{8} \times \frac{2}{7} \\ &= \frac{1}{21} \,. \end{aligned} \tag{11.4.23}$$

(ii) Note that the probability obtained in **(c)(i)** is the same probability as any other arrangement of the **same set** of colors.
Thus,

$$\begin{aligned} P(\text{one of each color}) = P(\textit{white} \to \textit{black} \to \textit{red}) \\ \times \\ \text{Number or permutations of (W, B, R)} \,. \end{aligned} \tag{11.4.24}$$

From Eq. (2.1.6), the number of permutations of the **set of 3** colors is,

$$\begin{aligned} {}^3P_3 &= \frac{3!}{(3-3)!} \\ &= \frac{3!}{1} \\ &= 6 \,. \end{aligned} \tag{11.4.25}$$

Hence, we can substitute in the above to find,

$$\begin{aligned} P(\text{one of each color}) &= P(\textit{white} \to \textit{black} \to \textit{red}) \times 6 \\ &= \frac{1}{21} \times 6 \\ &= \frac{2}{7} \,. \end{aligned} \tag{11.4.26}$$

(iii) We will split the marbles into 2 groups: White marbles (W) and not-White marbles (W'). In the setup, there are 4 W marbles and 5 W' marbles. The different combinations of the 3 marbles chosen, if a white is the last marble, are (W,W,W), (W'W,W), (W,W',W) and (W', W',W).
Using Eq. (2.1.1) and Eq. (2.1.2), we can find that,

$$\begin{aligned} P(W, W, W) &= \frac{4}{9} \times \frac{3}{8} \times \frac{2}{7} \\ &= \frac{1}{21} \,. \end{aligned} \tag{11.4.27}$$

$$\begin{aligned} P(W', W, W) &= \frac{5}{9} \times \frac{4}{8} \times \frac{3}{7} \\ &= \frac{5}{42} \,. \end{aligned} \tag{11.4.28}$$

$$\begin{aligned} P(W, W', W) &= \frac{4}{9} \times \frac{5}{8} \times \frac{3}{7} \\ &= \frac{5}{42}\,. \end{aligned} \tag{11.4.29}$$

$$\begin{aligned} P(W', W', W) &= \frac{5}{9} \times \frac{4}{8} \times \frac{4}{7} \\ &= \frac{10}{63}\,. \end{aligned} \tag{11.4.30}$$

$$\begin{aligned} P(\text{last is white}) &= P(W, W, W) + (W', W, W) + P(W, W', W) + P(W', W', W) \\ &= \frac{1}{21} + \frac{5}{42} + \frac{5}{42} + \frac{10}{63} \\ &= \frac{4}{9}\,. \end{aligned} \tag{11.4.31}$$

Alternatively, as we mentioned above, the order of the draws matters, up to an extent. You can convince yourself that the probability that the third marble is white, $P(\text{Anything}, \text{Anything}, W) = P(W, \text{Anything}, \text{Anything})$. Then, we simply need to compute the probability that the first marble is white,

$$\begin{aligned} P(\text{Anything}, \text{Anything}, W) &= P(W, \text{Anything}, \text{Anything}) \\ &= P(\text{1st marble is white}) \\ &= \frac{4}{9} \end{aligned} \tag{11.4.32}$$

Chapter 12

2012

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Chapter 13

2013

13.1 Module 1: Managing Uncertainty

1. **(a)** We can construct the following truth table for the proposition $\boldsymbol{(p \wedge q) \wedge \sim (p \vee q)}$.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \wedge q}$	$\boldsymbol{p \vee q}$	$\boldsymbol{\sim (p \vee q)}$	$\boldsymbol{(p \wedge q) \wedge \sim (p \vee q)}$
0	0	0	0	1	0
0	1	0	1	0	0
1	0	0	1	0	0
1	1	1	1	0	0

Table 13.1: Truth Table of $\boldsymbol{(p \wedge q) \wedge \sim (p \vee q)}$.

(b) From Def. 1.3.8, we see that $\boldsymbol{(p \wedge q) \wedge \sim (p \vee q)}$ is a contradiction since the truth value for the statement is always 0.

(c) Using De Morgan's Law (Def. 1.3.6),we can write a logically equivalent proposition as follows:

$$\boldsymbol{\sim (p \vee q) \equiv \sim p \wedge \sim q} \ . \tag{13.1.1}$$

(d) Let p="It is sunny" and let q="It is hot".
Therefore, the statement "It is not true that it is hot and sunny" can be expressed as the Boolean proposition $\boldsymbol{\sim (p \wedge q)}$.

Using De Morgan's Law (Def. 1.3.6) again,

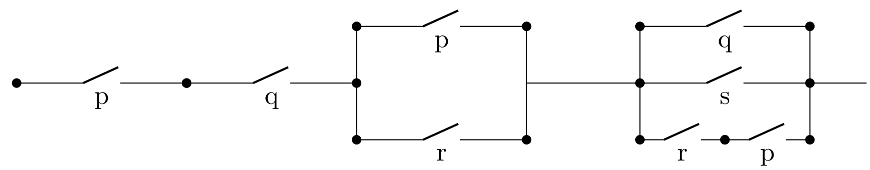
$$\boldsymbol{\sim (p \wedge q) \equiv \sim p \vee \sim q} \ . \tag{13.1.2}$$

Thus, the equivalent statement of $\boldsymbol{(\sim p \vee \sim q)}$ is "It is not hot or it is not sunny."

(e) **(i)** We can draw the switching circuit given by the Boolean expression

$$\boldsymbol{p \wedge q \wedge (p \vee r) \wedge (q \vee (r \wedge p) \vee s} \tag{13.1.3}$$

as follows.



(ii) Using the Commutative Law (Def. 1.3.8), we can manipulate the expression as follows,

$$\boldsymbol{p \wedge q \wedge (p \vee r) \wedge (q \vee (r \wedge p) \vee s) \equiv p \wedge (p \vee r) \wedge q \wedge (q \vee (r \wedge p) \vee s)}\,. \tag{13.1.4}$$

Let $\boldsymbol{A \equiv p \wedge (p \vee r)}$ and $\boldsymbol{B \equiv q \wedge (q \vee (r \wedge p) \vee s)}$.

Using the Absorptive Law (Def. 1.3.10), we can simplify $\boldsymbol{A}$ as,

$$\begin{aligned} \boldsymbol{A} &\equiv \boldsymbol{p \wedge (p \vee r)} \\ &\equiv \boldsymbol{p}\ . \end{aligned} \tag{13.1.5}$$

Similarly, using the Absorptive Law (Def. 1.3.10) twice, B can be simplified as,

$$\begin{aligned} \boldsymbol{B} &\equiv \boldsymbol{q \wedge (q \vee (r \wedge p) \vee s)} \\ &\equiv \boldsymbol{q \wedge (q \vee s)} \\ &\equiv \boldsymbol{q}\ . \end{aligned} \tag{13.1.6}$$

Thus, Eq. (13.1.4) can be simplified to,

$$\boldsymbol{p \wedge (p \vee r) \wedge q \wedge (q \vee (r \wedge p) \vee s) \equiv p \wedge q}\,. \tag{13.1.7}$$

(f) We can draw the following logic gates circuit to represent $\boldsymbol{\sim ((p \wedge q) \vee r)}$.

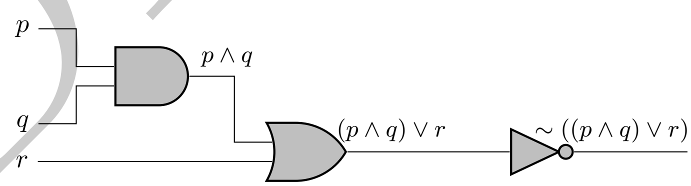


Figure 13.1: Logic Gate representation of $\boldsymbol{\sim ((p \wedge q) \vee r)}$.

2. (a) We can modify the given activity network to obtain the desired information.

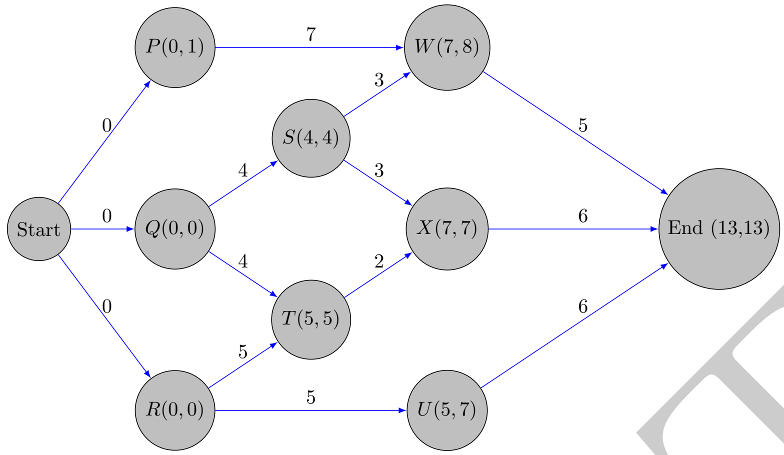


Figure 13.2: Activity network of the project.

(i) The earliest start time of S is 4 days. The earliest start time for X is 7 days.

(ii) The minimum completion time of the project if 13 days.

(iii) The latest start time of Q is 0 days. The latest start time of X is 7 days.

(iv) From Def. 1.2.7, there are three critical paths for the project. They are,

$$
\begin{aligned}
&\text{Start} \to Q \to S \to X \to \text{End}\,, \\
&\text{Start} \to Q \to T \to X \to \text{End}\,, \\
&\text{Start} \to R \to T \to X \to \text{End}\,. \qquad (13.2.1)
\end{aligned}
$$

(v) The float time of activity T is 0 days, using Def. 1.2.6.

(b) From Def. 1.2.4, we see that the degree of A is 4 and the degree of C is 2.

(c) **(i)** We can determine the supermarkets to which each warehouse must be assigned in order to minimise the cost of delivery by applying the Hungarian algorithm as follows.

$$\begin{array}{cccc} 6 & 5 & 7 & 9 \\ 2 & 6 & 5 & 8 \\ 10 & 5 & 1 & 9 \\ 11 & 6 & 3 & 8 \end{array}$$

Matrix From question

$$\begin{array}{cccc} 1 & 0 & 2 & 4 \\ 0 & 4 & 3 & 6 \\ 9 & 4 & 0 & 8 \\ 8 & 3 & 0 & 5 \end{array}$$

Reducing Rows

$$\begin{array}{cccc} 1 & 0 & 2 & 0 \\ 0 & 4 & 3 & 2 \\ 9 & 4 & 0 & 4 \\ 8 & 3 & 0 & 1 \end{array}$$

Reducing Columns

$$\begin{array}{cccc} 1 & 0 & 2 & 0 \\ 0 & 4 & 3 & 2 \\ 9 & 4 & 0 & 4 \\ 8 & 3 & 0 & 1 \end{array}$$

Shading 0's

$$\begin{array}{cccc} 1 & 0 & 4 & 0 \\ 0 & 4 & 5 & 2 \\ 8 & 3 & 0 & 3 \\ 7 & 2 & 0 & 0 \end{array}$$

Applying Step 5, 1.1.2

$$\begin{array}{cccc} 1 & 0 & 4 & 0 \\ 0 & 4 & 5 & 2 \\ 8 & 3 & 0 & 3 \\ 7 & 2 & 0 & 0 \end{array}$$

Shading 0's

Table 13.2: Steps of the Hungarian Algorithm.

From Table 13.2, the possible pairings of warehouses and supermarkets are as follows,

$$\begin{aligned} &W_1 \to S_2\,, S_4\,, \\ &W_2 \to S_1\,, \\ &W_3 \to S_3\,, \\ &W_4 \to S_3\,, S_4\,. \end{aligned} \tag{13.2.2}$$

Therefore, the optimum pairings to minimize the cost of delivery are,

$$\begin{aligned} &W_1 \to S_2\,, \\ &W_2 \to S_1\,, \\ &W_3 \to S_3\,, \\ &W_4 \to S_4\,. \end{aligned} \tag{13.2.3}$$

(ii) The total cost of transportation for the 4 warehouses is,

$$\text{Total cost} = 5 + 2 + 1 + 8 = 16\,\text{dollars}\,. \tag{13.2.4}$$

13.2 Module 2: Probability and Distributions

3. **(a)** We can use the properties of $E(X)$ and $\mathrm{Var}(X)$ in Section 2.12.

(i) We use the property that,

$$E[X+Y] = E[X] + E[Y]\,. \tag{13.3.1}$$

Substituting the given quantities,

$$\begin{aligned} E[X+Y] &= 5+7 \\ &= 12\,. \end{aligned} \tag{13.3.2}$$

(ii) We use the property that,

$$E[2X-3Y] = 2\times E[X] - 3\times E[Y]\,. \tag{13.3.3}$$

Substituting the given quantities,

$$\begin{aligned} E[2X-3Y] &= 2\times 5 - 3\times 7 \\ &= 10-21 \\ &= -11\,. \end{aligned} \tag{13.3.4}$$

(iii) We use the property that,

$$\mathrm{Var}[2X-3Y] = 2^2\times \mathrm{Var}[X] + 3^2\times \mathrm{Var}[Y] \tag{13.3.5}$$

Substituting the given quantities,

$$\begin{aligned} \mathrm{Var}[2X-3Y] &= 4\times 3 + 9\times 4 \\ &= 12+36 \\ &= 48\,. \end{aligned} \tag{13.3.6}$$

(b) It is given that,

$$X \sim \mathrm{Geo}\left(\frac{1}{3}\right). \tag{13.3.7}$$

(i) Since X is discrete, we can write,

$$P(X\geq 3) = P(X>2)\,. \tag{13.3.8}$$

Then, using Note 2.6.2,

$$P(X>2) = (1-p)^2\,. \tag{13.3.9}$$

Substituting, we find,

$$\begin{aligned} P(X\geq 3) &= \left(\frac{2}{3}\right)^2 \\ &= \frac{4}{9}\,. \end{aligned} \tag{13.3.10}$$

(ii) From Eq. (2.6.2),

$$E(X) = \frac{1}{p}\,. \tag{13.3.11}$$

Substituting, we find,

$$\begin{aligned} E(X) &= \frac{1}{\frac{1}{3}} \\ &= 3\,. \end{aligned} \tag{13.3.12}$$

(iii) From 2.6.1,

$$P(X = 5) = p \times q^{5-1}\,. \tag{13.3.13}$$

Substituting, we find

$$\begin{aligned} P(X = 5) &= \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^4 \\ &= \left(\frac{1}{3}\right)\left(\frac{16}{81}\right) \\ &= \frac{16}{243}\,. \end{aligned} \tag{13.3.14}$$

(c) We are given that X is normally distributed with parameters $\mu = 60$ and $\sigma = 10$.

$$X \sim N(60, 10^2)\,. \tag{13.3.15}$$

(i) In order to find $P(X \geq x)$, we must first standardize the random variable X to the normal random variable $Z = \frac{X-\mu}{\sigma}$. Thus, we get that,

$$\begin{aligned} P(X \geq 65) &= P\left(\frac{X-\mu}{\sigma} \geq \frac{65-\mu}{\sigma}\right) \\ &= P\left(Z \geq \frac{65-60}{10}\right) \\ &= P\left(Z \geq \frac{5}{10}\right) \\ &= P\left(Z \geq \frac{1}{2}\right) \\ &= 1 - \Phi\left(\frac{1}{2}\right) \\ &= 1 - 0.691 \\ &= 0.309\,. \end{aligned} \tag{13.3.16}$$

(ii) Similar to **(3)(c)(i)**, we get that,

$$\begin{aligned} P(40 \leq X \leq 80) &= P\left(\frac{40-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{80-\mu}{\sigma}\right) \\ &= P\left(\frac{40-60}{10} \leq Z \leq \frac{80-60}{10}\right) \\ &= P\left(\frac{-20}{10} \leq Z \leq \frac{20}{10}\right) \\ &= P(-2 \leq Z \leq 2) \\ &= \Phi(2) - \Phi(-2) \\ &= \Phi(2) - (1 - \Phi(2)) \\ &= 2 \times \Phi(2) - 1 \\ &= 2 \times (0.977) - 1 \\ &= 0.954\,. \end{aligned} \tag{13.3.17}$$

4. (a) Let X be the discrete random variable representing the number of accidents which occur on that highway per week. From the given information, we can express this as,

$$X \sim \text{Pois}(1)\,. \tag{13.4.1}$$

(i) Using Def. 2.7.1, we know that,

$$P(X = x) = \frac{1^x \times e^{-1}}{x!}\,. \tag{13.4.2}$$

Therefore, we can find,

$$\begin{aligned} P(X = 2) &= \frac{1^2 \times e^{-1}}{2!} \\ &= \frac{1 \times e^{-1}}{2} \\ &= \frac{1}{2e} \\ &= 0.184\,. \end{aligned} \tag{13.4.3}$$

(ii) Let Y be the discrete random variable representing the number of accidents which occur on that highway during a 4-week period. We can express this as

$$Y \sim \text{Pois}(4)\,. \tag{13.4.4}$$

Using Def. 2.7.1, we know that,

$$P(Y = y) = \frac{4^y \times e^{-4}}{y!}\,. \tag{13.4.5}$$

Since Y is discrete, we notice that,

$$\begin{aligned} P(Y \geq 4) &= 1 - P(Y \leq 3) \\ &= 1 - \left[P(Y = 0) + P(Y = 1) + P(Y = 2) + P(Y = 3)\right]\,. \end{aligned} \tag{13.4.6}$$

Using Eq. (13.4.5), we proceed by finding each of these,

$$\begin{aligned} P(Y = 0) &= \frac{4^0 \times e^{-4}}{0!} \\ &= \frac{1 \times e^{-4}}{1} \\ &= \frac{1}{e^4}\,, \end{aligned} \tag{13.4.7}$$

$$\begin{aligned} P(Y = 1) &= \frac{4^1 \times e^{-4}}{1!} \\ &= \frac{4 \times e^{-4}}{1} \\ &= \frac{4}{e^4}\,, \end{aligned} \tag{13.4.8}$$

$$\begin{aligned} P(Y=2) &= \frac{4^2 \times e^{-4}}{2!} \\ &= \frac{16 \times e^{-4}}{2} \\ &= \frac{8}{e^4}\,, \end{aligned} \tag{13.4.9}$$

and,

$$\begin{aligned} P(Y=3) &= \frac{4^3 \times e^{-4}}{3!} \\ &= \frac{64 \times e^{-4}}{6} \\ &= \frac{64}{6e^4}\,. \end{aligned} \tag{13.4.10}$$

We can therefore calculate $P(Y \leq 3)$ by substituting the above as follows,

$$\begin{aligned} P(Y \leq 3) &= P(Y=0) + P(Y=1) + P(Y=2) + P(Y=3) \\ &= \frac{1}{e^4} + \frac{4}{e^4} + \frac{8}{e^4} + \frac{64}{6e^4} \\ &= \frac{1}{e^4} \times \left(1 + 4 + 8 + \frac{64}{6}\right) \\ &= \frac{1}{e^4} \times \left(\frac{142}{6}\right) \\ &= \frac{142}{6e^4} \\ &= 0.433\,. \end{aligned} \tag{13.4.11}$$

Thus, we get that,

$$\begin{aligned} P(Y \geq 4) &= 1 - P(Y \leq 3) \\ &= 1 - 0.433 \\ &= 0.567\,. \end{aligned} \tag{13.4.12}$$

(iii) Let the discrete random variable Z be the number of 4-week periods in which at least 4 accidents occurred. From **(4)(a)(ii)**, we know that the probability that at least 4 accidents occur in a 4-week period is 0.567.
From this information, we can conclude that

$$Z \sim \text{Bin}(13, 0.567)\,, \tag{13.4.13}$$

and Def. 2.5.1, the probability mass function of Z, $P(Z=z)$, can be expressed as,

$$P(Z=z) = \binom{13}{z} (0.567)^z (1-0.567)^{13-z}\,. \tag{13.4.14}$$

Thus, we can find,

$$\begin{aligned} P(Z=11) &= \binom{13}{11}(0.567)^{11}(1-0.567)^{13-11} \\ &= \binom{13}{11}(0.567)^{11}(0.433)^2 \\ &= 78\times(0.567)^{11}\times(0.433)^2 \\ &= 0.028\,. \end{aligned} \tag{13.4.15}$$

(b) Let X be the discrete random variable representing the number of defective nails in a box of 100. At first,

$$X \sim \text{Bin}(100, 0.02)\,, \tag{13.4.16}$$

From Section 2.8, we can notice that,

$$n = 100 \geq 15 \text{ and } np = 100\times 0.02 = 2 < 15, \tag{13.4.17}$$

and so, X can be approximated as a Poisson Distribution as follows,

$$\begin{aligned} &X \sim \text{Pois}(np) \\ \Longrightarrow &X \sim \text{Pois}(2)\,. \end{aligned} \tag{13.4.18}$$

Using Def. 2.7.1, the probability mass function of X is,

$$P(X=x) = \frac{2^x\times e^{-2}}{x!}\,. \tag{13.4.19}$$

Since X is discrete, we know that,

$$P(X\leq 2) = P(X=0)+P(X=1)+P(X=2)\,. \tag{13.4.20}$$

Next, we calculate each probability,

$$\begin{aligned} P(Y=0) &= \frac{2^0\times e^{-2}}{0!} = \frac{1\times e^{-2}}{1} \\ &= \frac{1}{e^2}\,, \end{aligned} \tag{13.4.21}$$

$$\begin{aligned} P(Y=1) &= \frac{2^1\times e^{-2}}{1!} = \frac{2\times e^{-2}}{1} \\ &= \frac{2}{e^2}\,, \end{aligned} \tag{13.4.22}$$

and,

$$\begin{aligned} P(Y=2) &= \frac{2^2\times e^{-2}}{2!} = \frac{4\times e^{-2}}{2} \\ &= \frac{2}{e^2}\,, \end{aligned} \tag{13.4.23}$$

Substituting in the above, we find

$$\begin{aligned} P(X \le 2) &= P(X=0) + P(X=1) + P(X=2) \\ &= \frac{1}{e^2} + \frac{2}{e^2} + \frac{2}{e^2} \\ &= \frac{5}{e^2} \\ &= 0.677\,. \end{aligned} \tag{13.4.24}$$

(c) It is given that the continuous random variable, X, has a uniform distribution.

(i) From Prop. 2.9.1, we know that,

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1 \tag{13.4.25}$$

We must first split up the integral over the different regions,

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{0} f(x)\,\mathrm{d}x + \int_{0}^{3} f(x)\,\mathrm{d}x + \int_{3}^{\infty} f(x)\,\mathrm{d}x\,, \tag{13.4.26}$$

before we can substitute and evaluate,

$$\begin{aligned} 0 + \int_0^3 k\,\mathrm{d}x \ + 0 &= 1 \\ \left[\frac{kx}{1}\right]_0^3 &= 1 \\ [k(3)] - [k(0)] &= 1 \\ 3k &= 1 \\ \implies k &= \frac{1}{3}\,. \end{aligned} \tag{13.4.27}$$

Thus, we express $f(x)$ as,

$$f(x) = \begin{cases} \frac{1}{3} & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases} \tag{13.4.28}$$

(ii) We want to find the value of t such that $P(X > t) = 1/4$. Since X is continuous,

$$P(X > t) = \int_t^{\infty} f(x)\,\mathrm{d}x \tag{13.4.29}$$

Since $0 < P(X < t) < 1$, we know $0 < t < 3$, so the integral will split up as,

$$\int_t^{\infty} f(x)\,\mathrm{d}x = \int_t^3 f(x)\,\mathrm{d}x + \int_3^{\infty} f(x)\,\mathrm{d}x\,. \tag{13.4.30}$$

Evaluating,

$$\begin{aligned} P(X > t) &= \int_t^3 k \,\mathrm{d}x + \int_3^\infty 0 \,\mathrm{d}x \\ &= k\,[x]_t^3 \\ &= k[3 - t]\,. \end{aligned} \tag{13.4.31}$$

Equating to $1/4$, we can solve for t,

$$\begin{aligned} k(3 - t) &= \frac{1}{4} \\ t &= 3 - \frac{1}{4k} \\ &= \frac{9}{4} \end{aligned} \tag{13.4.32}$$

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Chapter 14

2014

14.1 Module 1: Managing Uncertainty

1. **(a)** Given the proposition $\boldsymbol{p} \implies \sim \boldsymbol{q}$, the contrapositive proposition is:

$$\boldsymbol{q} \implies \sim \boldsymbol{p}. \tag{14.1.1}$$

(b) We can construct the following truth table for the inverse of $\boldsymbol{p} \implies \sim \boldsymbol{q}$. Note that the inverse of $\boldsymbol{p} \implies \sim \boldsymbol{q}$ is $\sim \boldsymbol{p} \implies \boldsymbol{q}$.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\sim \boldsymbol{p}$	$\sim \boldsymbol{p} \implies \boldsymbol{q}$
0	0	1	1
0	1	1	0
1	0	0	1
1	1	0	1

Table 14.1: Truth Table of $\sim \boldsymbol{p} \implies \boldsymbol{q}$.

(c) **(i)** We can construct the following truth table for the proposition

$$(\boldsymbol{p} \implies \boldsymbol{q}) \vee (\boldsymbol{q} \implies \boldsymbol{r}) \tag{14.1.2}$$

as

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{p} \implies \boldsymbol{q}$	$\boldsymbol{q} \implies \boldsymbol{r}$	$(\boldsymbol{p} \implies \boldsymbol{q}) \vee (\boldsymbol{q} \implies \boldsymbol{r})$
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	1	0	1
0	1	1	1	1	1
1	0	0	0	1	1
1	0	1	0	1	1
1	1	0	1	0	1
1	1	1	1	1	1

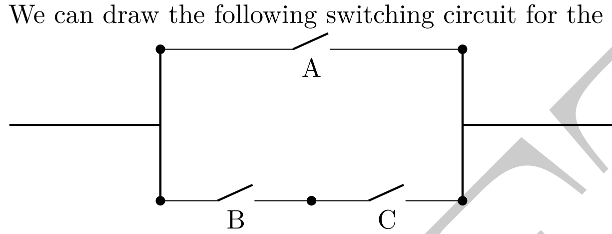
Table 14.2: Truth Table of $(\boldsymbol{p} \implies \boldsymbol{q}) \vee (\boldsymbol{q} \implies \boldsymbol{r})$.

(ii) From Def. 1.3.7, we can see that $(p \implies q) \vee (q \implies r)$ is a tautology because its truth value is always 1.

(d) We can write the following Boolean expression for the given logic gates circuit:

$$\sim (a \wedge b) \vee \sim c \ . \tag{14.1.3}$$

(e) **(i)** We can draw the following switching circuit for the proposition $A \vee (B \wedge C)$.



(ii) Using the Distributive Law (Def. 1.3.9), we see that,

$$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C) \, . \tag{14.1.4}$$

2. **(a)** **(i)** We can construct the following activity network from the given activities.

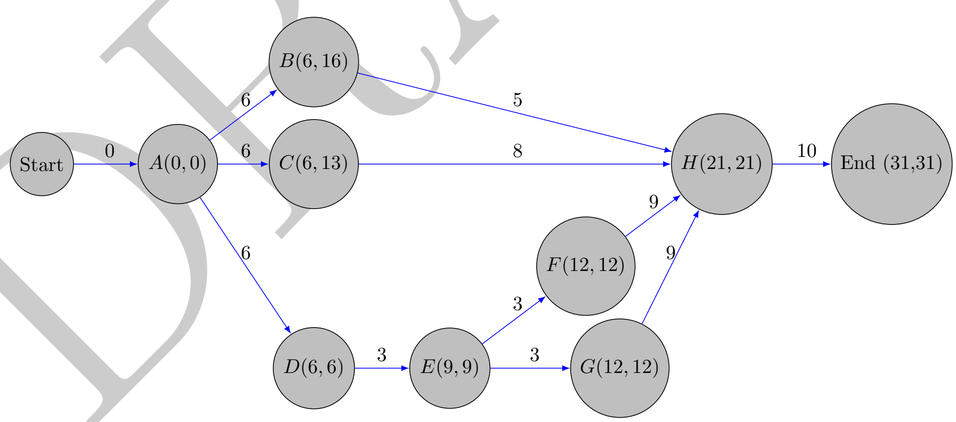


Figure 14.1: Activity Network of the operation.

(ii) We can complete the table as follows:

Activity	Earliest Start Time	Latest Start Time	Float Time
A	0	0	0
B	6	16	10
C	6	13	7
D	6	6	0
E	9	9	0
F	12	12	0
G	12	12	0
H	21	21	0

Table 14.3: Float Times of the activities.

(iii) Using Def. 1.2.7 and looking at Fig. 14.1 and Table 14.3, we can see that the two Critical Paths of this Activity Network are:

$$\text{Start} \to A \to D \to E \to F \to H \to \text{End}$$
$$\text{Start} \to A \to D \to E \to G \to H \to \text{End}$$

(b) **(i)** We can write the following Boolean expression for the given switching circuit:

$$\boldsymbol{p \wedge (p \vee \sim q)} \tag{14.2.1}$$

(ii) We can construct the truth table for the proposition $\boldsymbol{p \wedge (p \vee \sim q)}$

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{\sim q}$	$\boldsymbol{p \vee \sim q}$	$\boldsymbol{p \wedge (p \vee \sim q)}$
0	0	1	1	0
0	1	0	0	0
1	0	1	1	1
1	1	0	1	1

Table 14.4: Truth Table of $\boldsymbol{p \wedge (p \vee \sim q)}$.

Note: Although the question did not ask for it, this truth table is a proof of the Absorptive Law (Def. 1.3.10).

14.2 Module 2: Probability and Distributions

3. **(a)** First, note that,

$$P(A' \wedge B') = 1 - (P(A) + P(B) - P(A \wedge B)) \,. \tag{14.3.1}$$

From 2.2.9, we know that since A and B are independent,

$$P(A \wedge B) = P(A) \times P(B) \,. \tag{14.3.2}$$

Substituting this in the above, we find,

$$\begin{aligned} P(A' \wedge B') &= 1 - (P(A) + P(B) - P(A) \times P(B)) \\ &= 1 - (0.6 + 0.15 - (0.6 \times 0.15)) \\ &= 1 - (0.11) \\ &= 0.89\,. \end{aligned} \tag{14.3.3}$$

(b) We are given that 3 members of the choir are chosen for a special occasion.

(i) **a)** First, note that the total number of combinations that can be chosen is,

$$^{30}C_3 = \frac{30!}{27! \times 3!} = 4060\,. \tag{14.3.4}$$

To find the probability that two sopranos and one tenor is chosen, we must find the number of ways that this can happen. From Def. 2.1.1, we know that the number of ways to chose two sopranos and one tenor is,

$$\begin{aligned} \#(2 \text{ Sopranos and } 1 \text{ Tenor}) &= \#(2 \text{ Sopranos}) \times \#(1 \text{ Tenor}) \\ &= \,^{12}C_2 \times \,^{6}C_1 \\ &= \frac{12!}{10! \times 2!} \times \frac{6!}{6! \times 0!} \\ &= 66 \times 6 \\ &= 396\,. \end{aligned} \tag{14.3.5}$$

Thus, the probability that two sopranos and one tenor is chosen, $P(2S + 1T)$, is,

$$\begin{aligned} P(2S + 1T) &= \frac{396}{4060} \\ &= \frac{99}{1015}\,. \end{aligned} \tag{14.3.6}$$

b) The number of ways to chose 1 soprano, 1 tenor and 1 bass is,

$$\begin{aligned} \#(1 \text{ Soprano, } 1 \text{ Tenor and } 1 \text{ Bass}) &= \#(1 \text{ Soprano}) \times \#(1 \text{ Tenor}) \times \#(1 \text{ Bass}) \\ &= \,^{12}C_1 \times \,^{6}C_1 \times \,^{5}C_1 \\ &= 12 \times 6 \times 5 \\ &= 360\,. \end{aligned} \tag{14.3.7}$$

Thus, the probability that one soprano, one bass and one tenor is chosen, $P(1S+1T+1B)$, is,

$$\begin{aligned} P(1S + 1T + 1B) &= \frac{360}{4060} \\ &= \frac{18}{203}\,. \end{aligned} \tag{14.3.8}$$

c) From Def. 2.2.5, the probability that 3 tenors are chosen, given that all three parts are the same, is given as,

$$P(3\text{ Tenors}|3\text{ parts are the same}) = \frac{\text{Probability that 3 tenors are chosen}}{\text{Probability that all 3 parts are the same}}\,. \tag{14.3.9}$$

The probability that 3 tenors are chosen is given as,

$$\begin{aligned}\text{Probability of 3 tenors} &= \frac{{}^{6}C_3}{4060}\\ &= \frac{20}{4060}\\ &= \frac{1}{203}\,.\end{aligned} \tag{14.3.10}$$

Similarly, the probability that all 3 parts are the same is,

$$\begin{aligned}\text{Probability that 3 parts are the same} &= P(3S) + P(3A) + P(3T) + P(3B)\\ &= \frac{{}^{12}C_3 + {}^{7}C_3 + {}^{6}C_3 + {}^{5}C_3}{4060}\\ &= \frac{220 + 35 + 20 + 10}{203}\\ &= \frac{285}{4060}\\ &= \frac{57}{812}\,.\end{aligned} \tag{14.3.11}$$

Thus, we get that,

$$\begin{aligned}P(3\text{ Tenors}|3\text{ parts are the same}) &= \frac{\frac{1}{203}}{\frac{57}{812}}\\ &= \frac{4}{57}\,.\end{aligned} \tag{14.3.12}$$

(ii) We should first divide the parts into 3 groups: Bass(5), Tenor(6), NOT Bass or Tenor(19). In order to find the number of committees with exactly 2 basses and 3 tenors, we must use Def. 2.1.1 as follows,

$$\begin{aligned}\text{Committees of 9, with 2 Basses and 3 Tenors} &= {}^{5}C_2 \times {}^{6}C_3 \times {}^{19}C_4\\ &= 10 \times 20 \times 3876\\ &= 775200\,.\end{aligned} \tag{14.3.13}$$

Thus, the probability that the committee chosen has exactly 2 basses and three tenors is,

$$\begin{aligned}P(\text{Exactly 2B and 3T}) &= \frac{775200}{\text{Total number of committees}}\\ &= \frac{775200}{{}^{30}C_9}\\ &= \frac{775200}{14307150}\\ &\approx 0.054\,. \end{aligned} \tag{14.3.14}$$

(iii) Since all the singers are distinct, we can place the singers in a line instead on a circle to make the calculation easier. We will denote bassists with B_x where x is the number of possible bass singers in this particular position. Similarly, we will denote T_y where y is the number of possible tenors in this position. The $T_{^6C_2}$ represents the two tenors that are next to each other. The 2! represents the number of ways that the 2 tenors can be arranged.
The line can be formed as follows,

$$(2! \times T_{^6C_2}) : B_5 : T_4 : B_4 : T_3 : B_3 : T_2 : B_2 : T_1 : B_1\,. \tag{14.3.15}$$

This line can then be wrapped around the circular table with any starting position and still retain the arrangement. Using Def. 2.1.1, we can find the number of ways to arrange the line by multiplying all of the x and y values as follows,

$$\begin{aligned}\text{Number of Arrangements} &= 2! \times {}^6C_2 \times 5 \times 4 \times 4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1\\ &= 2 \times 15 \times 5 \times 4 \times 4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1\\ &= 86400\,. \end{aligned} \tag{14.3.16}$$

As we have arranged 10 groups in Eq. (14.3.15), we must divide the total number of arrangements by 10 in order to calculate the circular arrangements as follows,

$$\begin{aligned}\text{Number of Circular Arrangements} &= \frac{86400}{10}\\ &= 8640\,. \end{aligned} \tag{14.3.17}$$

4. (a) (i) Let X be the discrete random variable representing "the number of faults which occur in 15 meters of cloth",

$$X \sim \text{Pois}(3)\,. \tag{14.4.1}$$

From Section 2.7, we know that,

$$P(X = x) = \frac{3^x \times e^{-3}}{x!}\,.\,, \tag{14.4.2}$$

hence,

$$\begin{aligned}P(X = 4) &= \frac{3^4 \times e^{-3}}{4!}\\ &= \frac{81 \times e^{-3}}{24}\\ &= 0.168\,. \end{aligned} \tag{14.4.3}$$

(ii) Since we know that the rate of faults in 15 meters of cloth is equal to 3, the rate of faults in 60 meters of cloth is $\frac{60}{15} \times 3 = 12$.
Let Y be the discrete random variable representing "the number of faults which occur in 60 meters of cloth",

$$Y \sim \text{Pois}(12)\,. \tag{14.4.4}$$

From Section 2.7,

$$P(Y = y) = \frac{12^y \times e^{-12}}{y!}\,. \tag{14.4.5}$$

Since Y is discrete, we know that,

$$\begin{aligned} P(Y \geq 2) &= 1 - P(Y < 2) \\ &= 1 - \left(P(Y = 0) + P(Y = 1)\right). \end{aligned} \tag{14.4.6}$$

Thus, using Eq. (14.4.5), we can calculate,

$$\begin{aligned} P(Y \geq 2) &= 1 - \left(P(Y = 0) + P(Y = 1)\right) \\ &= 1 - \left(\frac{12^0 \times e^{-12}}{0!} + \frac{12^1 \times e^{-12}}{1!}\right) \\ &= 1 - \left(\frac{1 \times e^{-12}}{1} + \frac{12 \times e^{-12}}{1}\right) \\ &= 1 - \left(13 \times e^{-12}\right) \\ &= 1 - 0.00008 \\ &= 0.99992\,. \end{aligned} \tag{14.4.7}$$

(b) We are given that the mass of oranges, X, can be modeled by a normal distribution,

$$X \sim N(62.2, 3.6^2)\,. \tag{14.4.8}$$

(i) In order to find the probability that $X < 60$, we must first standardize the random variable, X, to the normal random variable, Z. From Section 2.10, we standardize and calculate as follows,

$$\begin{aligned}
P(X < 60) &= P\left(\frac{X-\mu}{\sigma} < \frac{60-\mu}{\sigma}\right) \\
&= P\left(Z < \frac{60-62.2}{3.6}\right) \\
&= P\left(Z < -\frac{2.2}{3.6}\right) \\
&= P\left(Z < -\frac{11}{18}\right) \\
&= \Phi\left(-\frac{11}{18}\right) \\
&= 1 - \Phi\left(\frac{11}{18}\right) \\
&= 1 - 0.729 \\
&= 0.271\,.
\end{aligned} \tag{14.4.9}$$

(ii) Similar to **(b)(i)**, we standardize as follows,

$$\begin{aligned}
P(61 < X < 64) &= P\left(\frac{61-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{64-\mu}{\sigma}\right) \\
&= P\left(\frac{61-62.2}{3.6} < Z < \frac{64-62.2}{3.6}\right) \\
&= P\left(-\frac{1.2}{3.6} < Z < \frac{1.8}{3.6}\right) \\
&= P\left(-\frac{1}{3} < Z < \frac{1}{2}\right) \\
&= \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{3}\right) \\
&= \Phi\left(\frac{1}{2}\right) - \left(1 - \Phi\left(\frac{1}{3}\right)\right) \\
&= 0.691 - (1 - 0.631) \\
&= 0.322\,.
\end{aligned} \tag{14.4.10}$$

(c) We are given the relevant probabilities of 2 independent random variables, X and Y.

(i) $(X + Y = 3)$ can happen in exactly 3 ways:

- $X = 0$ and $Y = 3$
- $X = 1$ and $Y = 2$
- $X = 2$ and $Y = 1$

Since X and Y are independent, following 2.2.4, we know that

$$P(X = x \wedge Y = y) = P(X = x) \times P(Y = y)\,. \tag{14.4.11}$$

We can therefore calculate $P(X+Y=3)$ as follows,

$$\begin{aligned} P(X+Y=3) &= P(X=0 \ \wedge \ Y=3) + P(X=1 \ \wedge \ Y=2) + P(X=2 \ \wedge \ Y=1) \\ &= [P(X=0)P(Y=3)] + [P(X=1)P(Y=2)] + [P(X=2)P(Y=1)] \\ &= [0.2 \times 0.25] + [0.3 \times 0.3] + [0.5 \times 0.1] \\ &= 0.05 + 0.075 + 0.05 \\ &= 0.175. \end{aligned} \tag{14.4.12}$$

(ii) **a)** From Def. 2.3.2,

$$E(X) = \sum_{\forall k} (x_k \times P(X = x_k)) \ . \tag{14.4.13}$$

Substituting, we find,

$$\begin{aligned} E(X) &= (0 \times 0.2) + (1 \times 0.3) + (2 \times 0.5) \\ &= 0 + 0.3 + 1 \\ &= 1.3. \end{aligned} \tag{14.4.14}$$

b) From Def. 2.3.4, we know,

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 \,. \tag{14.4.15}$$

Thus, we need to find $E(X^2)$. Using Def. 2.3.3,

$$E(X^2) = \sum_{\forall k} \left(x_k^2 \times P(X = x_k)\right) \ . \tag{14.4.16}$$

Substituting, we find

$$\begin{aligned} E(X^2) &= (0^2 \times 0.2) + (1^2 \times 0.3) + (2^2 \times 0.5) \\ &= (0 \times 0.2) + (1 \times 0.3) + (4 \times 0.5) \\ &= 0 + 0.3 + 2 \\ &= 2.3\,. \end{aligned} \tag{14.4.17}$$

Thus, we can find $\mathrm{Var}(X)$ by substituting the above,

$$\begin{aligned} \mathrm{Var}(X) &= E(X^2) - (E(X))^2 \\ &= 2.3 - 1.3^2 \\ &= 2.3 - 1.69 \\ &= 0.61. \end{aligned} \tag{14.4.18}$$

c) Repeating the above, can use Def. 2.3.2 as follows,

$$\begin{aligned} E(Y) &= \sum_{\forall k} (y_k \times P(Y = y_k)) \\ &= (0 \times 0.2) + (1 \times 0.1) + (2 \times 0.3) + (3 \times 0.25) + (4 \times 0.15) \\ &= 0 + 0.1 + 0.6 + 0.75 + 0.6 \\ &= 2.05. \end{aligned} \tag{14.4.19}$$

d) Again, following Def. 2.3.4, we know,

$$\mathrm{Var}(Y) = E(Y^2) - (E(Y))^2\,. \tag{14.4.20}$$

Using Def. 2.3.3, we get that,

$$\begin{aligned} E(Y^2) &= \sum_{\forall k} \left(y_k^2 \times P(Y = y_k)\right) \\ &= (0^2 \times 0.2) + (1^2 \times 0.1) + (2^2 \times 0.3) + (3^2 \times 0.25) + (4^2 \times 0.15) \\ &= (0 \times 0.2) + (1 \times 0.1) + (4 \times 0.3) + (9 \times 0.25) + (16 \times 0.15) \\ &= 0 + 0.1 + 1.2 + 2.25 + 2.4 \\ &= 5.95\,. \end{aligned} \tag{14.4.21}$$

And so, we can find $\mathrm{Var}(Y)$ as,

$$\begin{aligned} \mathrm{Var}(Y) &= E(Y^2) - (E(Y))^2 \\ &= 5.95 - 2.05^2 \\ &= 5.95 - 4.2025 \\ &= 1.7475. \end{aligned} \tag{14.4.22}$$

(iii) Using Section 2.12, we can calculate the following,

a) We use the property that,

$$E(3X - 2Y) = 3E(X) - 2E(Y)\,. \tag{14.4.23}$$

Substituting, we find,

$$\begin{aligned} E(3X - 2Y) &= 3 \times 1.3 - 2 \times 2.05 \\ &= 3.9 - 4.1 \\ &= -0.2. \end{aligned} \tag{14.4.24}$$

b) We use the property that,

$$\mathrm{Var}(3X - 2Y) = 3^2\mathrm{Var}(X) + 2^2\mathrm{Var}(Y)\,. \tag{14.4.25}$$

Substituting, we find,

$$\begin{aligned} \mathrm{Var}(3X - 2Y) &= 9 \times 0.61 + 4 \times 1.7475 \\ &= 5.49 + 6.99 \\ &= 12.48. \end{aligned} \tag{14.4.26}$$

Chapter 15

2015

15.1 Module 1: Managing Uncertainty

1. (a) Let x be the number of cases of glass bottles and let y be the number of cases of plastic bottles.

(i) a) The profit function to be maximized is,

$$P = 20x + 15y\,. \tag{15.1.1}$$

b) The inequalities in this problem are,

$$\begin{aligned} \text{Minimum constraint of cases of glass bottles: } & x \geq 0\,, \\ \text{Minimum constraint of cases of plastic bottles: } & y \geq 0\,, \\ \text{Time constraint on sterilising machine: } & 4x + 4y \leq 3000\,, \\ \text{Time constraint on capping machine: } & 3x + 6y \leq 3000\,. \end{aligned} \tag{15.1.2}$$

(ii) a) The graph of this linear programming problem is represented below.

b) The feasible region is shaded in Graph 15.1.

(iii) a) We can determine the number of each type of bottle required for maximum profit by performing a tour of the vertices:

Vertice	$P = 20x + 15y$
(0, 0)	0
(0, 500)	7500
(750, 0)	15000
(500, 250)	13750

Table 15.1: Tour of Vertices

To maximize the profit, 750 cases of glass bottles and 0 cases of plastic bottles should be manufactured.

b) From Table 15.1, the maximum profit is \$15000.

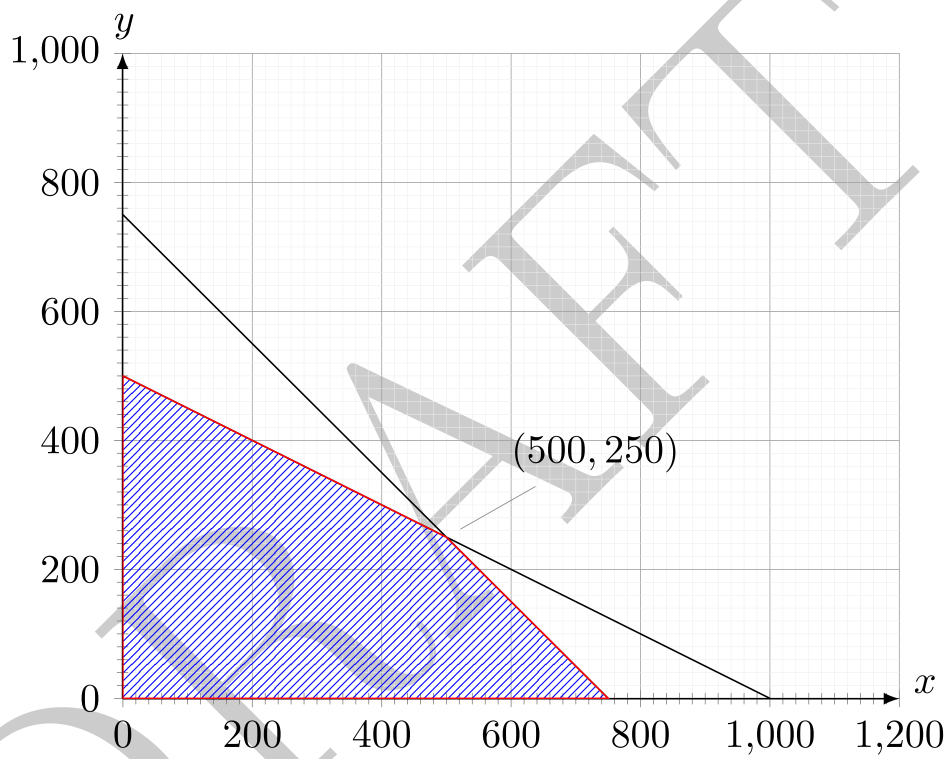


Figure 15.1: Linear Programming Graph.

(b) The shortest path from Town S to Town G is,

$$S \to C \to D \to F \to G \quad (20)\,. \tag{15.1.3}$$

The longest path from Town S to Town G is ,

$$S \to D \to F \to E \to G \quad (31)\,. \tag{15.1.4}$$

Note: The travel times between $A \to C$ and $C \to B$ were not filled in so we did not consider these connections.

2. **(a)** **(i)** We can determine the class to which each teacher must be assigned to minimise the total teaching time by applying the Hungarian Algorithm as follows.

$$\begin{array}{cccc} 30 & 32 & 31 & 34 \\ 35 & 33 & 30 & 30 \\ 29 & 31 & 28 & 33 \\ 32 & 34 & 29 & 32 \end{array} \qquad \begin{array}{cccc} 0 & 2 & 1 & 4 \\ 5 & 3 & 0 & 0 \\ 1 & 3 & 0 & 5 \\ 3 & 5 & 0 & 3 \end{array} \qquad \begin{array}{cccc} 0 & 0 & 1 & 4 \\ 5 & 1 & 0 & 0 \\ 1 & 1 & 0 & 5 \\ 3 & 3 & 0 & 3 \end{array}$$

Matrix From question — Reducing Rows — Reducing Columns

$$\begin{array}{cccc} 0 & 0 & 1 & 4 \\ 5 & 1 & 0 & 0 \\ 1 & 1 & 0 & 5 \\ 3 & 3 & 0 & 3 \end{array} \qquad \begin{array}{cccc} 0 & 0 & 3 & 4 \\ 5 & 1 & 2 & 0 \\ 0 & 0 & 0 & 4 \\ 2 & 2 & 0 & 2 \end{array} \qquad \begin{array}{cccc} 0 & 0 & 3 & 4 \\ 5 & 1 & 2 & 0 \\ 0 & 0 & 0 & 4 \\ 2 & 2 & 0 & 2 \end{array}$$

Shading 0's — Applying Step 5, 1.1.2 — Shading 0's

Table 15.2: Steps of the Hungarian Algorithm.

From Table 15.2, we can see that the possible matchings for the teachers are as follows:

$$\begin{aligned} &\text{Mrs Jones} \to \text{Class 1, Class 2}\,, \\ &\text{Mr James} \to \text{Class 4}\,, \\ &\text{Mrs Wright} \to \text{Class 1, Class 2, Class 3}\,, \\ &\text{Ms Small} \to \text{Class 3}\,. \end{aligned} \tag{15.2.1}$$

Therefore, the matchings to minimize the total time are,

$$\begin{aligned} &\text{Mrs Jones} \to \text{Class 1 or Class 2}\,, \\ &\text{Mr James} \to \text{Class 4}\,, \\ &\text{Mrs Wright} \to \text{Class 1 or Class 2}\,, \\ &\text{Ms Small} \to \text{Class 3}\,. \end{aligned} \tag{15.2.2}$$

(ii) The total time spent in the classroom by the four teachers can be found by summing the individual times of the teachers (based on the result of the Hungarian Algorithm). Evaluating this, we get

$$\text{Minimum time} = 30 + 30 + 31 + 29 = 120\,\text{minutes}\,. \tag{15.2.3}$$

(b) We can construct the following truth table for the proposition $(\boldsymbol{p} \wedge \boldsymbol{q}) \implies \boldsymbol{p}$.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p} \wedge \boldsymbol{q}$	$(\boldsymbol{p} \wedge \boldsymbol{q}) \implies \boldsymbol{p}$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	1

Table 15.3: Truth Table of $(\boldsymbol{p} \wedge \boldsymbol{q}) \implies \boldsymbol{p}$.

From Table 15.3 and using Def. 1.3.7, we can see that $(\boldsymbol{p} \wedge \boldsymbol{q}) \implies \boldsymbol{p}$, is a **tautology** as its truth value is always 1.

(c) **(i)** We can write the following Boolean expression for the given switching circuit:

$$\boldsymbol{a} \wedge (\boldsymbol{b} \vee (\boldsymbol{a} \wedge \boldsymbol{b})) \wedge (\boldsymbol{a} \vee (\sim \boldsymbol{a} \wedge \boldsymbol{b}))\,. \tag{15.2.4}$$

(ii) Using the Commutative Law (Def. 1.3.8), we can manipulate this expression as follows,

$$\boldsymbol{a} \wedge (\boldsymbol{b} \vee (\boldsymbol{a} \wedge \boldsymbol{b})) \wedge (\boldsymbol{a} \vee (\sim \boldsymbol{a} \wedge \boldsymbol{b})) \equiv \boldsymbol{a} \wedge (\boldsymbol{a} \vee (\sim \boldsymbol{a} \wedge \boldsymbol{b})) \wedge (\boldsymbol{b} \vee (\boldsymbol{a} \wedge \boldsymbol{b}))\,, \tag{15.2.5}$$

Let $\boldsymbol{P} \equiv \boldsymbol{a} \wedge (\boldsymbol{a} \vee (\sim \boldsymbol{a} \wedge \boldsymbol{b}))$ and let $\boldsymbol{Q} \equiv (\boldsymbol{b} \vee (\boldsymbol{a} \wedge \boldsymbol{b}))$.

Using the Distributive (Def. 1.3.9), Annulment (Def. 1.3.1) and the Absorptive (Def. 1.3.10) laws on P, we see that,

$$\begin{aligned} \boldsymbol{P} &\equiv \boldsymbol{a} \wedge (\boldsymbol{a} \vee (\sim \boldsymbol{a} \wedge \boldsymbol{b})) \\ &\equiv \boldsymbol{a} \wedge ((\boldsymbol{a} \vee \sim \boldsymbol{a}) \wedge (\boldsymbol{a} \vee \boldsymbol{b})) \\ &\equiv \boldsymbol{a} \wedge \boldsymbol{T} \wedge (\boldsymbol{a} \vee \boldsymbol{b}) \\ &\equiv \boldsymbol{a} \wedge (\boldsymbol{a} \vee \boldsymbol{b}) \\ &\equiv \boldsymbol{a}\,. \end{aligned} \tag{15.2.6}$$

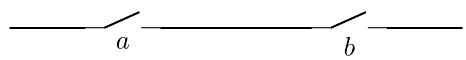
Similarly, using Commutative Law (Def. 1.3.8) and Absorptive Law (Def. 1.3.10) on Q, we see that,

$$\begin{aligned} \boldsymbol{Q} &\equiv (\boldsymbol{b} \vee (\boldsymbol{a} \wedge \boldsymbol{b})) \\ &\equiv (\boldsymbol{b} \vee (\boldsymbol{b} \wedge \boldsymbol{a})) \\ &\equiv \boldsymbol{b}\,. \end{aligned} \tag{15.2.7}$$

Therefore, Eq. (15.2.5) becomes,

$$\boldsymbol{a} \wedge (\boldsymbol{b} \vee (\boldsymbol{a} \wedge \boldsymbol{b})) \wedge (\boldsymbol{a} \vee (\sim \boldsymbol{a} \wedge \boldsymbol{b})) \equiv \boldsymbol{a} \wedge \boldsymbol{b}\,. \tag{15.2.8}$$

The expression, $\boldsymbol{a} \wedge \boldsymbol{b}$, can be expressed as a switching circuit as follows,



15.2 Module 2: Probability and Distributions

3. **(a)** Let the probability a bolt is defective, $P(D) = 0.2$.

(i) Let X be the discrete random variable representing "the number of defective bolts in a random sample of size 15". From Def. 2.5.1, we see that X can be modelled using a Binomial Distribution,

$$X \sim \text{Bin}(15, 0.2)\,, \tag{15.3.1}$$

and

$$P(X = x) = \binom{15}{x} 0.2^x (1 - 0.2)^{15-x}\,. \tag{15.3.2}$$

We want to find $P(X > 1)$. To solve this question, we can either find the sum of the different probabilities for defective bolts from 2 to 15 OR we can find the probability that the number of defective bolts is ≤ 1. This is because $P(X > 1) = 1 - P(X \leq 1)$.

Using the latter, we see that it is much simpler to calculate $P(X \leq 1)$ by noticing that

$$P(X \leq 1) = P(X = 1) + P(X = 0). \tag{15.3.3}$$

Solving for each probability, we get

$$\begin{aligned} P(X = 0) &= \binom{15}{0} 0.2^0 (1 - 0.2)^{15-0} \\ &= 1 \times 1 \times 0.8^{15} \\ &= 0.8^{15}\,, \end{aligned} \tag{15.3.4}$$

and,

$$\begin{aligned} P(X = 1) &= \binom{15}{1} 0.2^1 (1 - 0.2)^{15-1} \\ &= 15 \times 0.2 \times 0.8^{14} \\ &= 3 \times 0.8^{14}\,. \end{aligned} \tag{15.3.5}$$

Substituting, in the above we find,

$$\begin{aligned} P(X > 1) &= 1 - P(X \leq 1) \\ &= 1 - (P(X=0) + P(X=1)) \\ &= 1 - (0.8^{15} + 3 \times 0.8^{14}) \\ &= 1 - 0.167 \\ &= 0.833\,. \end{aligned} \tag{15.3.6}$$

(ii) We now want to find the smallest value of n such that,

$$\frac{\sqrt{\mathrm{Var}[X]}}{E[X]} < 0.1\,. \tag{15.3.7}$$

We can substitute the definitions for $E(X)$ and $\mathrm{Var}(X)$ using Eqs. 2.5.2 and Eq. 2.5.3,

$$\frac{\sqrt{np(1-p)}}{np} < 0.1 \tag{15.3.8}$$

$$\implies \sqrt{\frac{np(1-p)}{n^2p^2}} < 0.1. \tag{15.3.9}$$

$$\tag{15.3.10}$$

Simplifying this expression yields,

$$\sqrt{\frac{1-p}{np}} < 0.1\,. \tag{15.3.11}$$

Now, we evaluate the expression and solve for n,

$$\begin{aligned} &\implies \sqrt{\frac{0.8}{0.2n}} < 0.1 \\ &\implies \sqrt{\frac{4}{n}} < 0.1 \\ &\implies \frac{2}{\sqrt{n}} < 0.1 \\ &\implies \frac{2}{0.1} < \sqrt{n} \\ &\implies 20 < \sqrt{n}\,. \end{aligned} \tag{15.3.12}$$

Lastly, we square the expression to give

$$400 < n. \tag{15.3.13}$$

We get that n must be an integer that is greater than 400. Therefore, the smallest value of n that satisfies our given criteria is $n = 401$.

(b) In order to find $P(X \leq 2)$, we can use a Poisson approximation to the binomial distribution.

From Def. 2.8.1, we know that certain conditions must hold for a binomial distribution to be approximated by a Poisson distribution. From the given distribution, we can see that $n = 500 > 50$ and $np = 2.5 < 5$, and so the following approximation holds,

$$X \sim \text{Pois}(2.5)\,. \tag{15.3.14}$$

Now, we want to find $P(X \leq 2)$. For a Poisson Distribution,

$$P(X \leq 2) = P(0) + P(1) + P(2)\,. \tag{15.3.15}$$

By using Eq. 2.7.1, we can evaluate each of the probabilities,

$$\begin{aligned} P(X = 0) &= \frac{2.5^0 \times e^{-2.5}}{0!} \\ &= \frac{1 \times e^{-2.5}}{1} \\ &= e^{-2.5} \\ &= 0.082\,, \end{aligned} \tag{15.3.16}$$

$$\begin{aligned} P(X = 1) &= \frac{2.5^1 \times e^{-2.5}}{1!} \\ &= \frac{2.5 \times e^{-2.5}}{1} \\ &= 2.5 \times e^{-2.5} \\ &= 0.205\,, \end{aligned} \tag{15.3.17}$$

and,

$$\begin{aligned} P(X = 2) &= \frac{2.5^2 \times e^{-2.5}}{2!} \\ &= \frac{2.5^2 \times e^{-2.5}}{2} \\ &= 0.257\,. \end{aligned} \tag{15.3.18}$$

Finally, substituting for $P(X \leq 2)$, we get

$$\begin{aligned} P(X \leq 2) &= 0.082 + 0.205 + 0.257 \\ &= 0.544\,. \end{aligned} \tag{15.3.19}$$

(c) In order to find, $P(X \leq 25)$, we must use a normal approximation to the Poisson distribution as seen in Def 2.11.1.

Therefore, by Note 2.11.1 and using a continuity correction, we will find the probabilty that $P(X \leq 25.5)$.
Standardizing using Property (ii) in Note 2.10.1, we evaluate the probability as follows,

$$\begin{aligned} P(X \leq 25.5) &= P(Z < \frac{25.5 - 20}{\sqrt{20}}) \\ &= P(Z < 1.23) \\ &= 0.891\,. \end{aligned} \tag{15.3.20}$$

(d) To find $P(X < 4)$, we can either sum all the individual probabilities from $X = 1$ to 3 OR we can find $P(X > 3)$ because $P(X < 4) = 1 - P(X > 3)$.
Using the property of 2.6.2,

$$P(X > 3) = 0.7^3\,, \tag{15.3.21}$$

we can determine $P(X < 4)$ as follows,

$$\begin{aligned} P(X < 4) &= 1 - P(X > 3) \\ &= 1 - 0.7^3 \\ &= 1 - 0.343 \\ &= 0.657\,. \end{aligned} \tag{15.3.22}$$

4. (a) (i) The probability distribution function of a continuous random variable X is given, as well as $F(0.5) = 0.8$.

We want to determine the two unknowns a and b. We can use two constraints, that from Prop. 2.9.1,

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1 \tag{15.4.1}$$

and,

$$F(0.5) = 0.08\,. \tag{15.4.2}$$

For the first constraint, we must split up the integral over the different regions,

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{0} f(x)\,\mathrm{d}x + \int_{0}^{1} f(x)\,\mathrm{d}x + \int_{1}^{\infty} f(x)\,\mathrm{d}x \tag{15.4.3}$$

Substituting,

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = & \int_{-\infty}^{0} 0\,\mathrm{d}x + \int_{0}^{1} (a + bx)\,\mathrm{d}x + \int_{1}^{\infty} 0\,\mathrm{d}x \\ &= 0 + \int_{0}^{1} (a + bx)\,\mathrm{d}x \;+ 0 \\ &= \left[ax + \frac{bx^2}{2}\right]_0^1 \\ &= \left[a(1) + \frac{b(1)^2}{2}\right] - \left[a(0) + \frac{b(0)^2}{2}\right] \\ &= a + \frac{b}{2}\,. \end{aligned} \tag{15.4.4}$$

and so

$$a + \frac{b}{2} = 1 \tag{15.4.5}$$

In order to solve for a and b, we must formulate another equation using the cumulative distribution function of X given as $F(z)$. From Def. 2.9.4,

$$F(z) = \int_{-\infty}^{z} f(x)\,\mathrm{d}x \tag{15.4.6}$$

Since we will use $z = 0.5$, we must split up the integral as

$$F(z) = \int_{-\infty}^{0} f(x)\mathrm{d}x + \int_{0}^{z} f(x)\,\mathrm{d}x\,, \tag{15.4.7}$$

before we can substitute and evaluate,

$$\begin{aligned} F(z) &= 0 + \int_0^z (a + bx)dx \\ &= \left[ax + \frac{bx^2}{2}\right]_0^z \\ &= \left[a(z) + \frac{b(z)^2}{2}\right] - \left[a(0) + \frac{b(0)^2}{2}\right] \\ &= az + \frac{bz^2}{2}\,. \end{aligned} \tag{15.4.8}$$

Since we are given that $F(0.5) = 0.8$, our second equation becomes,

$$\begin{aligned} F(0.5) = a(0.5) + \frac{b(0.5)^2}{2} &= 0.8, \\ \Longrightarrow 0.5 \times a + 0.25 \times \frac{b}{2} &= 0.8, \\ \Longrightarrow \frac{a}{2} + \frac{b}{8} &= \frac{4}{5}. \end{aligned} \tag{15.4.9}$$

We will manipulate Eq. (15.4.9) by multiplying by 4 to obtain,

$$2a + \frac{b}{2} = \frac{16}{5}. \tag{15.4.10}$$

We can now solve for a by subtracting Eq. (15.4.10) from Eq. (15.4.5).

$$\begin{aligned} \left[2a + \frac{b}{2}\right] - \left[a + \frac{b}{2}\right] &= \left[\frac{16}{5}\right] - [1] \\ a &= \frac{11}{5} \\ &= 2.2. \end{aligned} \tag{15.4.11}$$

Finally, substituting $a = 2.2$ into Eq. (15.4.5), we can solve for b as follows,

$$\begin{aligned} 2.2 + \frac{b}{2} &= 1 \\ \frac{b}{2} &= 1 - 2.2 \\ &= -1.2 \\ \implies b &= -1.2 \times 2 \\ &= -2.4 \end{aligned} \tag{15.4.12}$$

Thus, $a = 2.2$ and $b = -2.4$.

(ii) From Def. 2.9.4, we can solve for $P(X > 0.5)$ as follows,

$$\begin{aligned} P(X > 0.5) &= 1 - F(0.5) \\ &= 1 - 0.8 \\ &= 0.2. \end{aligned} \tag{15.4.13}$$

(b) **(i)** It is given that X is normally distributed as $X \sim N(9, 1^2)$. The values of m and n can be given as,

$$\begin{aligned} m &= \text{Total} \times P(8 \leq X < 9) \\ &= 400 \times P(8 \leq X < 9). \end{aligned} \tag{15.4.14}$$

$$\begin{aligned} n &= \text{Total} \times P(X \geq 10) \\ &= 400 \times P(X \geq 10). \end{aligned} \tag{15.4.15}$$

To calculate $P(8 \leq X < 9)$, we must first standardize the random variable X, to the normal random variable Z. From Section 2.10, we standardize and calculate as follows,

$$\begin{aligned} P(8 \leq X < 9) &= P\left(\frac{8-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} < \frac{9-\mu}{\sigma}\right) \\ &= P\left(\frac{8-9}{1} \leq Z < \frac{9-9}{1}\right) \\ &= P\left(-1 \leq Z < 0\right) \\ &= \Phi(0) - \Phi(-1) \\ &= \Phi(0) - (1 - \Phi(1)) \\ &= \Phi(0) + \Phi(1) - 1 \\ &= 0.5 + 0.841 - 1 \\ &= 0.341\,. \end{aligned} \tag{15.4.16}$$

Similarly, in order to calculate $P(X \geq 10)$, we have to standardize in a similar manner as Eq. (15.4.16). This result follows as,

$$\begin{aligned} P(X \geq 10) &= P\left(\frac{X-\mu}{\sigma} \geq \frac{10-\mu}{\sigma}\right) \\ &= P\left(Z \geq \frac{10-9}{1}\right) \\ &= P\left(Z \geq \frac{1}{1}\right) \\ &= P\left(Z \geq 1\right) \\ &= 1 - \Phi(1) \\ &= 1 - 0.841 \\ &= 0.159\,. \end{aligned} \tag{15.4.17}$$

Using Eq. (15.4.16) and Eq. (15.4.17), we can calculate the values of m and n from Eq. (15.4.15) as follows,

$$\begin{aligned} m &= 400 \times P(8 \leq X < 9) \\ &= 400 \times 0.341 \\ &= 136.5\,. \end{aligned} \tag{15.4.18}$$

$$\begin{aligned} n &= 400 \times P(X \geq 10) \\ &= 400 \times 0.159 \\ &= 63.5\,. \end{aligned} \tag{15.4.19}$$

Therefore, $m = 136.5$ and $n = 63.5$.

However, we can also notice that X is assumed to have a normal distribution. This means that the cumulative probabilities are symmetric about the mean of the distribution (which in this case is 9). Therefore, $P(8 \leq X < 9) = P(9 \leq X < 10)$ and $P(X < 8) = P(X \geq 10)$. This method spares us from the calculation above and provides the answer quickly.

(ii) We can state 2 hypotheses:

- H_0 : The data can be modeled by a normal distribution with mean 9 and standard deviation 1.
- H_1 : The data cannot be modeled by a normal distribution with mean 9 and standard deviation 1.

We are asked to perform a χ^2 goodness-of-fit test at the 5% significance level. From Section(placeholder for CHI Squared notes) (**TODO**), we know that, **TODO:** Insert ref

$$\chi^2_{\text{calc.}} = \sum_{\forall k}\left(\frac{(O_k - E_k)^2}{E_k}\right)\,. \tag{15.4.20}$$

We will formulate a table and obtain the relevant values. See Table 15.4.

X	Observed, O	Expected, E	(O-E)	$(\text{O-E})^2$	$\frac{(O-E)^2}{E}$
$x < 8$	80	63.5	16.5	272.25	4.29
$8 \leq x < 9$	180	136.5	43.5	1892.25	13.86
$9 \leq x < 10$	50	136.5	-86.5	7482.25	54.82
$x \geq 10$	90	63.5	26.5	702.25	11.06
$\chi^2_{\text{calc.}}$					84.03

Table 15.4: Observed and Expected Values.

We will reject the null hypothesis, H_0, iff

$$\chi^2_{calc.} > \chi^2_{\alpha}(\nu)\,,$$

where α is the significance level, and υ is the number of degrees of freedom.

We know $\alpha = 0.05$ and $\nu = 4 - 1 = 3$. Therefore, the value of the test statistic, $\chi^2_{0.05}(3) = 7.815$.

Since we have found that $\chi^2_{\text{calc.}} = 84.03$, which is definitely greater than $\chi^2_{0.05}(3) = 7.815$, we reject H_0. Thus, there is not sufficient evidence at the 5% significance level to suggest that the data can be modeled by a normal distribution with mean 9 and standard deviation 1.

Chapter 16

2016

16.1 Module 1: Managing Uncertainty

1. **(a)** **(i)** We wish to show that the the proposition $(\boldsymbol{p} \wedge \sim \boldsymbol{q}) \vee (\sim \boldsymbol{p} \wedge \boldsymbol{q})$ is equivalent to proposition $(\boldsymbol{p} \vee \boldsymbol{q}) \wedge (\sim \boldsymbol{p} \vee \sim \boldsymbol{q})$. We can construct the following truth tables to determine whether these propositions are equivalent.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\sim \boldsymbol{p}$	$\sim \boldsymbol{q}$	$(\boldsymbol{p} \wedge \sim \boldsymbol{q})$	$(\sim \boldsymbol{p} \wedge \boldsymbol{q})$	$(\boldsymbol{p} \wedge \sim \boldsymbol{q}) \vee (\sim \boldsymbol{p} \wedge \boldsymbol{q})$
0	0	1	1	0	0	0
0	1	1	0	0	1	1
1	0	0	1	1	0	1
1	1	0	0	0	0	0

Table 16.1: Truth Table of $(\boldsymbol{p} \wedge \sim \boldsymbol{q}) \vee (\sim \boldsymbol{p} \wedge \boldsymbol{q})$.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\sim \boldsymbol{p}$	$\sim \boldsymbol{q}$	$(\boldsymbol{p} \vee \boldsymbol{q})$	$(\sim \boldsymbol{p} \vee \sim \boldsymbol{q})$	$(\boldsymbol{p} \vee \boldsymbol{q}) \wedge (\sim \boldsymbol{p} \vee \sim \boldsymbol{q})$
0	0	1	1	0	1	0
0	1	1	0	1	1	1
1	0	0	1	1	1	1
1	1	0	0	1	0	0

Table 16.2: Truth Table of $(\boldsymbol{p} \vee \boldsymbol{q}) \wedge (\sim \boldsymbol{p} \vee \sim \boldsymbol{q})$.

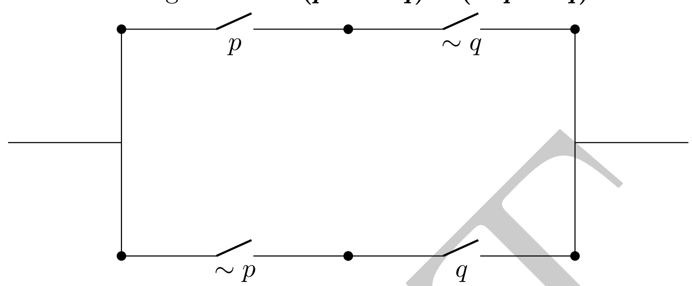
(ii) Using the Distributive (Def. 1.3.9) law three times, the Complement law (Def. 1.3.4) and the Identity law (Def. 1.3.2), we can see that,

$$\begin{aligned}(\boldsymbol{p} \wedge \sim \boldsymbol{q}) \vee (\sim \boldsymbol{p} \wedge \boldsymbol{q}) &\equiv ((\boldsymbol{p} \wedge \sim \boldsymbol{q}) \vee \sim \boldsymbol{p}) \wedge ((\boldsymbol{p} \wedge \sim \boldsymbol{q}) \vee \boldsymbol{q}) \\ &\equiv ((\boldsymbol{p} \vee \sim \boldsymbol{p}) \wedge (\sim \boldsymbol{q} \vee \sim \boldsymbol{p})) \wedge ((\boldsymbol{p} \vee \boldsymbol{q}) \wedge (\sim \boldsymbol{q} \vee \boldsymbol{q})) \\ &\equiv \boldsymbol{T} \wedge (\sim \boldsymbol{q} \vee \sim \boldsymbol{p}) \wedge (\boldsymbol{q} \vee \boldsymbol{p}) \wedge \boldsymbol{T} \\ &\equiv (\boldsymbol{q} \vee \sim \boldsymbol{p}) \wedge (\boldsymbol{q} \vee \boldsymbol{p})\,. \qquad (16.1.1)\end{aligned}$$

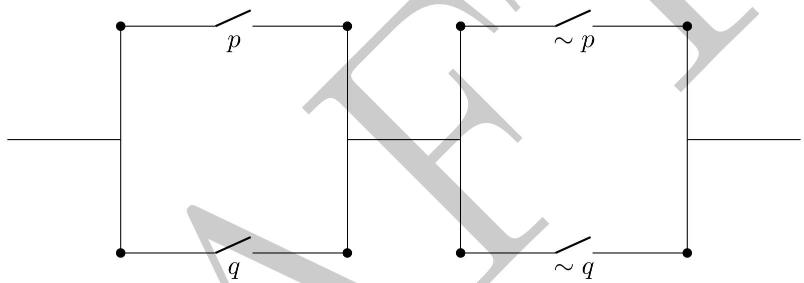
Therefore,

$$(\boldsymbol{p} \wedge \sim \boldsymbol{q}) \vee (\sim \boldsymbol{p} \wedge \boldsymbol{q}) \equiv (\sim \boldsymbol{p} \vee \sim \boldsymbol{q}) \wedge (\boldsymbol{q} \vee \boldsymbol{p})\,. \tag{16.1.2}$$

(b) **(i)** We can draw the switching circuit of $(\boldsymbol{p} \wedge \sim \boldsymbol{q}) \vee (\sim \boldsymbol{p} \ \wedge \boldsymbol{q})$ below.



(ii) We can draw the switching circuit of $(\boldsymbol{p} \vee \boldsymbol{q}) \wedge (\sim \boldsymbol{p} \vee \sim \boldsymbol{q})$ below.



(c) We can write the following Boolean expression to represent the output of the given switching circuit.

$$(\boldsymbol{p} \wedge \sim \boldsymbol{q}) \vee (\boldsymbol{p} \wedge \sim \boldsymbol{r})\,. \tag{16.1.3}$$

(d) Given the following propositions,
$\boldsymbol{p}$: A person who eats red meat
$\boldsymbol{q}$: A person who has a high cholesterol reading
$\boldsymbol{r}$: A person with a normal cholesterol reading
$\boldsymbol{s}$: A person who suffers a heart attack

(i) The Boolean expression for "If a person eats red meat then that person may have a high cholesterol reading or suffer a heart attack." can be expressed as,

$$\boldsymbol{p} \implies (\boldsymbol{q} \vee \boldsymbol{s})\,. \tag{16.1.4}$$

(ii) The Boolean expression for "If a person does not suffer a heart attack then that person has a normal cholesterol reading and does not eat red meat." can be expressed as,

$$\sim \boldsymbol{s} \implies (\boldsymbol{r} \wedge \sim \boldsymbol{p})\ . \tag{16.1.5}$$

(iii) We can create the truth table for the proposition $\boldsymbol{p} \implies (\boldsymbol{q} \vee \boldsymbol{s})$.
From Table 16.3 and Def. 1.3.7, the propsition $\boldsymbol{p} \implies (\boldsymbol{q} \vee \boldsymbol{s})$ is not a tautology as its value is not always 1.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{s}$	$\boldsymbol{q \vee s}$	$\boldsymbol{p \implies (q \vee s)}$
0	0	0	0	1
0	0	1	1	1
0	1	0	1	1
0	1	1	1	1
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

Table 16.3: Truth Table of $\boldsymbol{p \implies (q \vee s)}$.

2. **(a)** We must maximize the expression, $P = 6x + 4y$, given the following inequalities:

$$\begin{aligned} 4x + 6y &\leq 48\,, \\ 4x + 2y &\leq 32\,, \\ 0 \leq y &\leq 5\,, \\ x &\geq 0\,. \end{aligned} \tag{16.2.1}$$

(i) We can draw the inequalities for the linear programming question below.

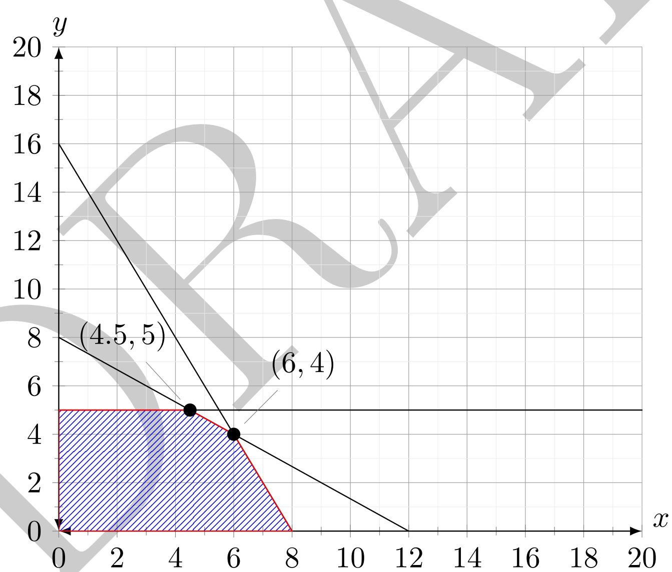


Figure 16.1: Linear Programming Graph.

(ii) The feasible region is shaded in Graph 16.1.

(iii) From Def. **??**, we can calculate P by performaing a tour of the vertices,

Vertice	$P = 6x + 4y$
(0, 0)	0
(0, 5)	20
(4.5, 5)	47
(6, 4)	52
(8, 0)	48

Table 16.4: Tour of Vertices

From Table 16.4, we see that the maximum value of P is 52, when $x = 6$ and $y = 4$.

(b) (i) We wish to assign four drivers to take passengers to four towns in order to minimise travel time. We can use the Hungarian algorithm on the data given.

$$\begin{matrix} 23 & 31 & 25 & 28 \\ 30 & 20 & 33 & 29 \\ 20 & 25 & 28 & 25 \\ 35 & 19 & 20 & 30 \end{matrix}$$

Matrix From question

$$\begin{matrix} 0 & 8 & 2 & 5 \\ 10 & 0 & 13 & 9 \\ 0 & 5 & 8 & 5 \\ 16 & 0 & 1 & 11 \end{matrix}$$

Reducing Rows

$$\begin{matrix} 0 & 8 & 1 & 0 \\ 10 & 0 & 12 & 4 \\ 0 & 5 & 7 & 0 \\ 16 & 0 & 0 & 6 \end{matrix}$$

Reducing Columns

$$\begin{matrix} 0 & 8 & 1 & 0 \\ 10 & 0 & 12 & 4 \\ 0 & 5 & 7 & 0 \\ 16 & 0 & 0 & 6 \end{matrix}$$

Shading 0's

Table 16.5: Showing the steps of the Hungarian Algorithm.

From Table 16.5, the possible matchings of the passengers and drivers are,

$$\begin{aligned} &A \to R, P\,, \\ &B \to H\,, \\ &C \to R, P\,, \\ &D \to H, S\,. \end{aligned} \tag{16.2.2}$$

Thus, the matchings that minimize the total travel time is,

$$\begin{aligned} &A \to R \text{ or } P\,, \\ &B \to H\,, \\ &C \to R \text{ or } P\,, \\ &D \to S\,. \end{aligned} \tag{16.2.3}$$

(ii) Therefore, the minimum total travel time is,

$$\text{Minimum Time} = 23 + 20 + 25 + 20 = 88\,\text{minutes}\,. \tag{16.2.4}$$

16.2 Module 2: Probability and Distributions

3. (a) (i) In total, there are 17 paintings available. Thus, using Def. 2.1.4, the number of portfolios that can be created is,

$$\begin{aligned} {}^{17}C_{12} &= \frac{17!}{(17-12)! \times 12!} \\ &= \frac{17!}{5! \times 12!} \\ &= 6188\,. \end{aligned} \tag{16.3.1}$$

(ii) In order to find the number of portfolios with 8 water-color and 4 oil paintings, we must first find the number of ways to choose 8 water-color paintings from 10 paintings and the number of ways to choose 4 oil paintings from 7 paintings.

For the 8 water-color paintings,

$$\begin{aligned} {}^{10}C_{8} &= \frac{10!}{(10-8)! \times 8!} \\ &= \frac{10!}{2! \times 8!} \\ &= 45\,. \end{aligned} \tag{16.3.2}$$

For the 4 oil paintings,

$$\begin{aligned} {}^{7}C_{4} &= \frac{7!}{(7-4)! \times 4!} \\ &= \frac{7!}{3! \times 4!} \\ &= 35\,. \end{aligned} \tag{16.3.3}$$

Therefore, using Def. 2.1.1, the number of portfolios with 8 water-color **and** 4 oil paintings is,

$$\begin{aligned} {}^{10}C_{8} \ \times \ {}^{7}C_{4} &= 45 \times 35 \\ &= 1575\,. \end{aligned} \tag{16.3.4}$$

(iii) Given that the selection of paintings are random, we can see that,

$$\begin{aligned} P(\text{8 water-color, 4 oil paintings}) &= \frac{\text{Portfolios with 8 water-color, 4 oil paintings}}{\text{Total number of portfolios}} \\ &= \frac{1575}{6188} \\ &= \frac{225}{884} \\ &= 0.255\,. \end{aligned} \tag{16.3.5}$$

(b) We know that the probability of rolling a 6 on a fair die is $\frac{1}{6}$. Let X be the discrete random variable representing "the number of rolls a player does before starting a game, up to and including the 6".

From this, we know that X follows a Geometric Distribution with probability of success, $p = \frac{1}{6}$, as follows

$$X \sim \text{Geo}\left(\frac{1}{6}\right). \tag{16.3.6}$$

(i) From Def. 2.6.1, the probability mass function, $P(X = x)$ is,

$$\begin{aligned} P(X = x) &= \frac{1}{6} \times \left(1 - \frac{1}{6}\right)^{x-1} \\ &= \frac{1}{6} \times \left(\frac{5}{6}\right)^{x-1} \end{aligned} \tag{16.3.7}$$

Thus, $P(X = 3)$ can be calculated as,

$$\begin{aligned} P(X = 3) &= \frac{1}{6} \times \left(\frac{5}{6}\right)^{3-1} \\ &= \frac{1}{6} \times \left(\frac{5}{6}\right)^{2} \\ &= \frac{1}{6} \times \left(\frac{25}{36}\right) \\ &= \frac{25}{216}. \end{aligned} \tag{16.3.8}$$

(ii) The probability that **at most** 5 rolls are necessary to start the game is the same as $P(X < 6)$. Since X is a discrete random variable,

$$P(X < 6) = 1 - P(X > 5). \tag{16.3.9}$$

From Eq. (2.6.2), we know that,

$$\begin{aligned} P(X > 5) &= \left(1 - \frac{1}{6}\right)^{5} \\ &= \left(\frac{5}{6}\right)^{5} \\ &= \frac{3125}{7776}. \end{aligned} \tag{16.3.10}$$

Therefore,

$$\begin{aligned} P(X < 6) &= 1 - P(X > 5) \\ &= 1 - \frac{3125}{7776} \\ &= \frac{4651}{7776}. \end{aligned} \tag{16.3.11}$$

(iii) From Def. 2.6.2,

$$\begin{aligned} E(X) &= \frac{1}{p} \\ &= \frac{1}{\frac{1}{6}} \\ &= 6\,. \end{aligned} \tag{16.3.12}$$

(c) Let X be the discrete the random variable representing the number of faulty reports per day. X follows a Poisson Distribution as,

$$X \sim \text{Pois}(2.5)\,. \tag{16.3.13}$$

(i) From Def. 2.7.1, we know that the probability mass function, $P(X = x)$ is,

$$P(X = x) = \frac{2.5^x \times e^{-2.5}}{x!}\,. \tag{16.3.14}$$

Thus, we can calculate $P(X = 4)$ as follows,

$$\begin{aligned} P(X = 4) &= \frac{2.5^4 \times e^{-2.5}}{4!} \\ &= \frac{39.065 \times e^{-2.5}}{24} \\ &= \frac{39.065}{24 \times e^{2.5}} \\ &= 0.134\,. \end{aligned} \tag{16.3.15}$$

(ii) Over a five-day period, the average rate will be 5 times the daily rate ($5 \times 2.5 = 12.5$). If we let Y represent the number of faulty reports per five-day period, we see that Y follows a Poisson Distribution as,

$$Y \sim \text{Pois}(12.5)\,. \tag{16.3.16}$$

Similarly, the probability mass function, $P(Y = y)$ is

$$P(Y = y) = \frac{12.5^y \times e^{-12.5}}{y!}\,. \tag{16.3.17}$$

Hence, we use this as follows,

$$\begin{aligned} P(Y = 5) &= \frac{12.5^5 \times e^{-12.5}}{5!} \\ &= \frac{12.5^5 \times e^{-12.5}}{120} \\ &= \frac{12.5^5}{120 \times e^{12.5}} \\ &= 0.00948\,. \end{aligned} \tag{16.3.18}$$

4. **(a)** We need to perform a χ^2 test at the 5% level of significance.

(i) The hypotheses can be given as

- H_0: "The data is equally distributed."
- H_1: "The data is not equally distributed."

(ii) A frequency table is given and we want to find whether there is an equal distribution of grades. Therefore, we get that our Expected Value, E, for all of the Grades is 20. We get,

Grade	Observed,O	Expected,O	$\frac{(O-E)^2}{E}$
A	14	20	1.8
B	22	20	0.2
C	30	20	5
D	18	20	0.2
F	16	20	0.8
Total, $\chi^2_{\text{calc.}}$			8

Table 16.6: χ^2 Observed and Expected Values.

Since none of the $\frac{(O-E)^2}{E}$ values is less than 5, we get that the number of degrees of freedom, ν, is,

$$\nu = 5 - 1 = 4\,. \tag{16.4.1}$$

Thus, the critical value for this test will be,

$$\chi^2_{0.05}(\nu) = \chi^2_{0.05}(4) = 9.488\,. \tag{16.4.2}$$

We will therefore reject H_0 if,

$$\chi^2_{\text{calc.}} > 9.488\,. \tag{16.4.3}$$

(iii) From Table 16.6, we found that the test statistic is equal to 8.

(iv) From Eq. (16.4.3), we see that $\left(\chi^2_{\text{calc.}} = 8\right)$ is not greater than $\left(\chi^2_{0.05}(4) = 9.488\right)$. Thus, we do not reject H_0.

(b) We are given the probability density function, f, of the continuous random variable X.

(i) We use Def. 2.9.1,

$$\int_{\infty}^{\infty} f(x)\,\mathrm{d}x = 1\,. \tag{16.4.4}$$

Splitting up the integral over the different regions,

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{0} f(x)\,\mathrm{d}x + \int_{0}^{3} f(x)\mathrm{d}x + \int_{3}^{\infty} f(x)\,\mathrm{d}x = 1\,, \tag{16.4.5}$$

we can then substitute and evaluate as,

$$\begin{aligned}
\int_{\infty}^{\infty} f(x)\,\mathrm{d}x &= 0 + \int_0^3 (k(4-x))\,\mathrm{d}x + 0 \\
&= \left[4kx - \frac{kx^2}{2}\right]_0^3 \\
&= \left[4k(3) - \frac{k(3)^2}{2}\right] - \left[4k(0) - \frac{k(0)^2}{2}\right] \\
&= 12k - \frac{9k}{2}\,. && (16.4.6)
\end{aligned}$$

Solving for k,

$$\begin{aligned}
k\left(12 - \frac{9}{2}\right) &= 1 \\
k &= \frac{1}{(12-\frac{9}{2})} \\
k &= \frac{1}{(\frac{15}{2})} \\
\implies k &= \frac{2}{15}\,. && (16.4.7)
\end{aligned}$$

(ii) Using Note 2.9.1,

$$P(X > 1) = \int_1^{\infty} f(x)\,\mathrm{d}x\,. \qquad (16.4.8)$$

Splitting up the integral in the appropriate regions, we have,

$$P(X > 1) = \int_1^3 f(x)\,\mathrm{d}x + \int_3^{\infty} f(x)\,\mathrm{d}x\,. \qquad (16.4.9)$$

Now, substituting $f(x)$ in these regions allows us to evaluate as follows

$$\begin{aligned}
P(1 < X) &= \int_1^3 \left(\frac{2}{15}(4-x)\right)\,\mathrm{d}x + 0 \\
&= \frac{2}{15} \times \int_1^3 (4-x)\,\mathrm{d}x \\
&= \frac{2}{15} \times \left[4x - \frac{x^2}{2}\right]_1^3 \\
&= \frac{2}{15} \times \left(\left[4(3) - \frac{(3)^2}{2}\right] - \left[4(1) - \frac{(1)^2}{2}\right]\right) \\
&= \frac{2}{15} \times \left(\left[\frac{(15}{2}\right] - \left[\frac{7}{2}\right]\right) \\
&= \frac{2}{15} \times \frac{8}{2} \\
&= \frac{8}{15}\,. && (16.4.10)
\end{aligned}$$

(iii) From, Def. 2.9.2, we note,

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x\,. \tag{16.4.11}$$

Again, we must split up the integral in the appropriate regions,

$$E(X) = \int_{-\infty}^{0} x f(x)\,\mathrm{d}x + \int_{0}^{3} x f(x)\,\mathrm{d}x + \int_{3}^{\infty} x f(x)\,\mathrm{d}x\,. \tag{16.4.12}$$

Substituting $f(x)$ in the regions, we find that,

$$\begin{aligned}
E(X) &= 0 + \int_0^3 \left(x \times \left(\frac{2}{15}(4-x) \right) \right) \mathrm{d}x + 0 \\
&= \frac{2}{15} \times \int_0^3 \left(4x - x^2\right) \mathrm{d}x \\
&= \frac{2}{15} \times \left[2x^2 - \frac{x^3}{3} \right]_0^3 \\
&= \frac{2}{15} \times \left(\left[2(3)^2 - \frac{(3)^3}{3} \right] - \left[2(0)^2 - \frac{(0)^3}{3} \right] \right) \\
&= \frac{2}{15} \times 9 \\
&= \frac{18}{15}\,.
\end{aligned} \tag{16.4.13}$$

(iv) From Note 2.9.1,

$$F(x) = \int_{-\infty}^{x} f(a)\,\mathrm{d}a \tag{16.4.14}$$

Now, we must consider the three possibilities, that $x \le 0$, $0 < x \le 3$ and $x > 3$. For $x \le 0$, we simply have,

$$\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(a)\,\mathrm{d}a \\
&= \int_{-\infty}^{x} 0\,\mathrm{d}a \\
&= 0\,.
\end{aligned} \tag{16.4.15}$$

For $0 < x \leq 3$, we have,

$$\begin{aligned}
F(x) &= \int_{-\infty}^{0} f(a)\,\mathrm{d}a + \int_{0}^{x} f(a)\,\mathrm{d}a \\
&= \int_{-\infty}^{0} 0\,\mathrm{d}x + \int_{0}^{x} k(4-x) \\
&= \int_{0}^{x} \left(\frac{2}{15}(4-a)\right)\,\mathrm{d}a \\
&= \frac{2}{15} \times \left[4a - \frac{a^2}{2}\right]_0^x \\
&= \frac{8x}{15} - \frac{x^2}{15}\,. \qquad (16.4.16)
\end{aligned}$$

Finally, for $x > 3$,

$$\begin{aligned}
F(x) &= \int_{-\infty}^{0} f(a)\,\mathrm{d}a + \int_{0}^{3} f(a)\,\mathrm{d}a + \int_{3}^{x} f(a)\,\mathrm{d}a \\
&= 0 + 1 + 0 \\
&= 1\,. \qquad (16.4.17)
\end{aligned}$$

Summarizing,

$$F(x) = \begin{cases} 0, & x \leq 0 \\ \frac{8x}{15} - \frac{x^2}{15}, & 0 \leq x \leq 3 \\ 1, & x \geq 3\,. \end{cases} \qquad (16.4.18)$$

(v) We can recall from Note 2.9.3,

$$P(1.5 < X < 2) = F(2) - F(1.5)\,. \qquad (16.4.19)$$

Substituting for F, we can evaluate as,

$$\begin{aligned}
P(1.5 < X < 2) &= \left(\frac{8(2)}{15} - \frac{(2)^2}{15}\right) - \left(\frac{8(1.5)}{15} - \frac{(1.5)^2}{15}\right) \\
&= \left(\frac{16}{15} - \frac{4}{15}\right) - \left(\frac{12}{15} - \frac{2.25}{15}\right) \\
&= \left(\frac{12}{15}\right) - \left(\frac{9.75}{15}\right) \\
&= \frac{2.25}{15} \\
&= 0.15\,. \qquad (16.4.20)
\end{aligned}$$

Part IV

Appendix

Bibliography

[1] Caribbean Examinations Council. *Applied Mathematics CAPE ®Past Papers*. Macmillan Education, 2015.

[2] Caribbean Examinations Council. *Applied Mathematics CAPE ®Past Papers*. Macmillan Education, 2016.